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ABSTRACT

This document is a revised edition of the 1967 Guidelines to Mathematics, K-6, and is thought to reveal few changes in detail and no changes of substance. The guidelines are seen as firmly rooted in the successful curriculums of the past while encouraging changes appropriate to the needs of students who will be adults in the computer age. Major sections include: (1) Mathematics Instruction: A Point of View; (2) Key Mathematical Content Objectives and Related Instructional Objectives; (3) Problem Solving; (4) Adjusting Mathematics Programs to Student Abilities; (5) Evaluation and Testing; (6) Criteria for Program Development; (7) An Agenda for Action: Recommendations for School Mathematics of the 1980's (NCTM); (8) Position Paper on Basic Skills; and (9) Bibliography. The document lists instructional objectives separately for grades kindergarten through grade eight. Each grade has objectives listed under the following headings, where appropriate: Sets and Numbers; Numeration Systems; Order; Number Systems, Operations and Their Properties; Ratio and Proportion; Computation; Size and Shape; Sets of Points; Symmetry; Congruence; Similarity; Constructions; and Coordinate Systems and Graphs. (MP)

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GUIDELINES TO MATHEMATICS K-8

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Wisconsin Department of Public Instruction / Herbert J. Grover, State Superintendent

GUIDELINES TO MATHEMATICS

K-8

Second Edition

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Bulletin No. 2126

FOREWORD TO THE SECOND EDITION

A careful comparison of this revised edition with the first edition, now fourteen years old, will reveal few changes in detail and no changes of substance. The authors of the 1967 *Guidelines to Mathematics, K-6* and the subsequent *Guidelines to Mathematics, 6-8* prepared, in the midst of the so-called "new math" era, a document which has remained contemporary through the "basic skills" era and anticipates the coming of the "problem solving" era.

No revolution in mathematics is advocated here. The revised guidelines are firmly rooted in the successful curriculums of the past while encouraging changes appropriate to the needs of students who will be adults in the computer age. A massive retraining of teachers is not called for, nor is a mathematics program which is incomprehensible to administrators, parents and taxpayers.

These revised guidelines include the deletion of a few previous objectives, and the introduction of a few others. References to set operations and set notation have been largely eliminated, as have numeration systems in bases other than ten. Calculations which are greatly simplified by using calculators (such as ordering fractions, divisibility tests, and computations with large numbers) have also been eliminated, as well as objectives dealing with Roman numerals. All these topics remain valuable sources of enrichment activities.

Newly introduced objectives include an expanded list of statistics objectives introduced at an earlier age, and computer literacy objectives.

The National Council of Teachers of Mathematics report, *An Agenda For Action: Recommendations for School Mathematics of the 1980s*, has been reproduced in this volume. The recommendation that "The mathematics curriculum should be organized around problem solving" is not explicitly incorporated into the "Key Mathematical Content Objectives and Related Instructional Objectives." Any mathematical knowledge or skill which students acquire which does not result in improved problem solving ability is inconsistent with the spirit, if not the letter, of these guidelines.

Herbert J. Grover
State Superintendent
September 1981

FOREWORD TO THE FIRST EDITION

In a dynamic society such as ours, change is constant. People accept this fact more readily in many fields than in the area of mathematics - particularly at the elementary school level. And yet, in recent years new projects, new courses, and new materials remind us that a great potential exists for improved changes in the elementary mathematics program.

The primary purpose of *Guidelines to Mathematics, K-8* is to help those who have responsibility in local school districts for providing a well conceived elementary mathematics program. Each of the sections of this guide has been designed to meet specific objectives. Although no section represents the final word on the topic being discussed, it is intended that all sections represent definitive statements which can serve as a sound foundation for further study and investigation.

The main section of this guide has been devoted to a careful development of the major concepts of elementary school mathematics and associated behavioral objectives. In addition, several other sections have been devoted to issues concerning the effective teaching of mathematics at the elementary school level.

The Wisconsin State Department of Public Instruction trusts that the efforts of all who helped make this publication possible will result in improved mathematical experiences for all Wisconsin youth.

William C. Kahl
State Superintendent
1967

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Curriculum documents such as *Guidelines to Mathematics, K-8*, are achieved only through the generous contributions of time and energy of many experts representing all aspects of the mathematics education community in Wisconsin. We acknowledge our debt to the following persons who assisted in the preparation of the first edition.

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Donald L. Chambers
September 1981

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Mathematics Instruction: A Point of View

The variety of mathematics programs now available for schools promises to benefit both teachers and students alike. Teachers have a greater opportunity than ever to select the materials and teaching techniques best suited to a student's needs. The basis used for selecting methods and materials is important and is dependent upon the viewpoint of the teacher in regard to the instructional task. Just as mathematics programs have changed, so might certain aspects of the traditional view of instruction be changed.

It has been common practice heretofore to consider mathematics skills and concepts as being "all-or-nothing" attainments. That is, mastery of a certain body of facts or skills has been traditionally regarded as the proper province of the teacher at a particular grade level. The failure of some students to achieve the expected level of proficiency has forced the teacher of the next grade to "reteach" these areas, a task that has long been a source of irritation to many teachers. This type of compartmentalization of a course of study should be minimized, if not eliminated. Instead of expecting boys and girls at a given grade level to master in a rather final sense a certain portion of the mathematical training, educators should develop an approach which recognizes the existence of individual growth rates and stresses the continuity of the instructional process.

TEACHING FOR GROWTH

Perhaps the one factor most essential to the success of the mathematics curriculum is an emphasis on understanding, that is, understanding of mathematics. This emphasis represents a marked shift in the focus of educators' attention from overt behavioral skills and social applications to understandings developed as a person organizes and codifies mathematical ideas. It is no longer deemed sufficient for a teacher to be satisfied with competent computational performance by a student in spite of the obvious necessity for such skill. The need to find a more efficient, a more enjoyable, and a more illuminating method of instruction has led to the consensus that clear and penetrating understanding of certain essential mathematics must precede, but certainly not supplant the traditional point of emphasis, computation. How is this to be achieved?

EXPERIENCING THE PHYSICAL WORLD

Certain principles of instruction deserve renewed emphasis. It is agreed that one should traverse from the known to the unknown; from the particular to the general; from the concrete to the abstract. Clearly, students should be encouraged to develop many concepts of mathematics from their experiences with physical objects.

This approach implies, in many cases, that prior to the introduction of a concept, time should be provided for each student to experiment with physical objects appropriate to the objective. For instance, every student should have the experience of comparing volumes of various three-dimensional figures such as cones, cylinders, spheres and prisms by using sand or liquids as a prelude to volume problems; should have "pre-number" experiences, such as learning to compare the numerousness of groups through the process of one-to-one matching, rather than counting in the traditional fashion; should have many experiences with locating cities on maps, and playing various games on coordinate charts, before graphing solution sets of linear equations.

By such purposeful "playing" important concepts that have been left largely to chance development will be given appropriate attention and will be established on a firm foundation.

USEFUL UNIFYING CONCEPTS

Because understanding is an ambiguous term and because a teacher must make as clear as possible the relationship between former objectives and present points of emphasis, it may be well to consider just a few of the many concepts that are especially important to clarify and develop from the earliest stages.

For the foreseeable future, the study of number systems will continue to be the essence of the mathematics program. Involved in this study are the collections of various kinds of number ideas, such as natural numbers, *together with* the operations defined on these number ideas and the properties of these operations (for example, commutative, associative, distributive properties). Obviously, the development of an understanding of such ideas must take place over an extended period of time and at considerably different rates for different students. There is no such thing as complete understanding of any particular number system by a student. Rather, teachers at each level should ascertain that the instruction is directed toward deepening and extending the broad mathematical streams cited in later sections of this report. How can this be done?

To grow mathematically, the student must be guided toward the acquisition of the broad and essential concepts which tie together seemingly disconnected particulars into a coherent general structure. The same operations are encountered in several peculiar and different contexts in succeeding grades. Each particular use really represents a different operation.

For instance, the multiplication of two natural numbers such as 3×5 may be properly regarded as the process of starting with nothing and adding 5 objects to a set 3 times. But can $1/2 \times 1/3$ be similarly interpreted as the process of repeatedly adding 1/3 objects for 1/2 times? Or what meaning can be ascribed to $(-3)(-5)$? Certainly not -5 things repeatedly added for -3 times! Such statements require different interpretations because the operation called *multiplication* has several meanings and definitions depending upon the kinds of numbers or set elements to which it is applied.

Natural numbers, integers, and rational numbers each combine somewhat differently under a given operation. But rather than invent new symbols to represent the changing definitions of an operation as it applies to the various number systems, one uses the same symbol throughout. In this situation, as in many similar ones, students must be taught to interpret symbols in context. Only in this way will mathematics shorthand serve the important purpose of clarifying and improving communication. Furthermore, the student should be led to understand that although the specific interpretation of an operation differs from one system to the next, a structural unity still remains which is founded on the commutative, associative, and distributive properties as they pertain to the operations. Because of this structural unity, one can rely on the context in which a symbol is used to determine its meaning, rather than create new symbols for each particular variation of a concept.

Again, the need for techniques which lead the child to discover ideas through experiences with physical objects must be stressed. Furthermore, attention must be given to rather subtle ideas that may have been overlooked or thought too obvious to mention in the past. For instance, when combining two sets of objects into a single collection, a child must come to the realization that the abstract property of numerousness is conserved: that there are exactly as many objects after as there were before the two groups were joined. As fundamental as such an assumption is to understanding the addition operation, it is by no means obvious to all beginners. This fact is substantiated by the work of Piaget and others.

Another matter often treated too lightly involves the freedoms or restrictions which may or may not exist for a given operation. When matching the elements of a set with the word names of the ordinal numbers, one must know that the order in which the objects are matched to the numerals will not affect the count. On the other hand, cases exist in which the order of presentation of information is important. One cannot mix up numerator and denominator with impunity or name the coordinates of a point by whim. The basis for free choice, where it exists, and the reasons for restrictions, as they occur, must be made clear if understanding is to be achieved.

To sum up, students should come to realize that mathematics is a logical system *which exists as a human invention*, formulated, enriched, extended, and revised in response to the twin needs to perfect it as a logical structure and to use it as a convenient method of describing certain aspects of nature as seen by man.

UNDUE EMPHASIS ON PARTICULARS

Educators should avoid giving undue emphasis to any single aspect of the total mathematics program. The idea of sets, for instance, should not be glorified beyond its usefulness in contributing to the attainment of the broad goals of the program. Similarly, it is unwise to go to the extreme of downgrading the importance of computational proficiency to the point where long range goals are placed in jeopardy. Furthermore, big words or impressive terms and symbols must never interfere with a student's understanding of the concepts which the words or symbols represent. Clarity of communication is the chief purpose of mathematical language and symbolism. If a particular term or symbol does not serve the cause of clarity, then it should not be used. Students, of course, must eventually learn how to use the language effectively and understand it in context. However, teachers should exercise great care in working toward such a goal and should avoid any proliferation of symbols or premature verbalizations that will hinder goal attainment.

PROBLEM SOLVING

One of the greatest challenges for the teacher is the development of appropriate, real-life problems which meaningfully involve learners. Classroom teachers recognize the limitations of textbooks in providing such verbal problems. Many problems fail to challenge students; they fail to represent a world which is real to young people, and too often they fail to contribute to the student's skill in problem solving situations outside the textbook. Notwithstanding these limitations, textbook verbal problems will continue to be prominent in most classrooms until individual teachers identify other ways of meeting the problem solving obligation.

There is increasing evidence that through experimental programs and through the efforts of individual classroom teachers, improved ways of meeting this obligation are being developed. Much is being heard about the importance of relating mathematics to other areas, of the use of computers in mathematics instruction, of units of study in mathematics, of enrichment activities in mathematics, of the study of the history of mathematics, and of students creating their own mathematics problems. Instead of stressing the social development of the student, these efforts emphasize the structure of mathematics through problem solving experiences. These activities are hopeful signs pointing to a greater stress being placed, in the near future, on the importance of meaningfully-structured problem solving experiences as a part of the mathematics program.

INDIVIDUAL DIFFERENCES

If mathematics instruction is viewed as a process of initiating understanding and of carefully nurturing this understanding as the student matures, it will then be necessary to discover effective techniques for accommodating the widely differing rates at which young people develop. Implicit in this statement is the need for schools and teachers to recognize that some students may not be able to grow substantially or may seem to terminate their potential for growth at some point along the way. In such cases, provision should be made for experiences most appropriate to the individual's welfare.

To date, no satisfactory method has been developed to cope with the vast range of individual differences. Flexible grouping procedures, nongraded classrooms, individualized instruction, team-teaching, computer-assisted instruction, and television teaching are receiving extensive testing and evaluation. Perhaps the experiments currently underway will yield effective techniques once more is learned about the problem. In the meantime, each teacher must exercise professional judgment and common sense in adopting an optimal arrangement that is compatible with the teacher's own abilities, with the characteristics of students in the class, and with the physical facilities and administrative policies of the school.

In summary, mathematics instruction should be viewed as a continuous effort to develop in the individual a knowledge of mathematics that is characterized by its depth and connectedness. To the extent possible, the student should be encouraged to experiment with the objects of the environment. Thus prepared, students may be led to the invention or discovery of those ideas which provide both a broad basis for further exploration and a sense of delight in a well-founded mastery of the subject. The task is a challenging one and deserves much effort. To this end, the teacher should do everything possible to see that instruction is well-planned and is provided on a regular basis. As an additional, but essential measure, cooperative action such as inservice education should be taken by school personnel to develop in the teaching staff a view of instruction appropriate to present day needs and opportunities.

Key Mathematical Content Objectives

and

Related Instructional Objectives

In the following outline, an attempt has been made to point out where key content objectives of mathematics might be introduced and developed in grades K-8. The corresponding instructional objectives suggest a sequence of development of these mathematical ideas which provides for their reinforcement and continuity throughout the K-8 mathematics program. By means of a sequence of development, key ideas can be extended to conform with the maturity and background experience of the student.

The suggested content objectives have been organized under eighteen main topics: Sets and Numbers; Numeration Systems; Order; Number Systems, Operations and Their Properties; Ratio and Proportion; Computation; Size and Shape; Sets of Points; Symmetry; Congruence; Similarity; Constructions; Coordinate Systems and Graphs; Measurement; Mathematical Sentences; Ordered Pairs, Relations and Functions; Statistics and Probability; and Computers.

It is not intended that the placement of topics in this outline be considered as the only correct arrangement or that all of the topics necessarily be taught at every grade level as presented here. The ideas listed for each grade level should be regarded as a *suggested* guide for *introducing* various topics; the outline is not intended to be all-inclusive. The teacher will find it necessary to alter the order of topics to meet the needs of the children or the needs of a particular group in a given classroom. Objectives introduced at any grade level should be reviewed, reinforced, and extended periodically throughout the subsequent grades.

Furthermore, no fixed amount of time or emphasis has been suggested for any objective in this outline. A disproportionate amount of space has been devoted to some topics for purposes of clarity, and the amount of space devoted should not be considered an indication of the relative importance of a topic. Many of the geometric objectives can be attained in a relatively short time as compared with the time and emphasis necessary to attain those objectives central to an understanding and use of the rational number system.

This outline should prove useful to the school administration, the mathematics curriculum committee, the teachers of a school district, and university instructors, in one or more of the following ways:

- As an orientation to the key content and instructional objectives of the K-8 mathematics curriculum.
- As a source of knowledge of the related objectives of school mathematics, their development and reinforcement throughout the entire K-8 mathematics program.
- As a means of evaluating the content and instructional objectives of the present mathematics program of the school.
- As a guide for the preparation of a mathematics curriculum guide, K-8, for use by the school district.
- As an aid in the evaluation and selection of textbooks.
- As a guide in determining the concepts that need to be highlighted in inservice activities for teachers.
- As an aid in choosing standardized tests.

Mathematical Concepts

SETS AND NUMBERS

A set is a collection of things. Sets may have many properties, but a property peculiar to a collection or set is that of *number*. When an orderly arrangement of these numbers has been made, one has the tools needed for counting. Numbers can be used in the cardinal sense, that is, to indicate numerousness. For example, a triangle has *three* vertices. They can also be used in the ordinal sense, that is, to indicate the arrangement of numbers by order. For example, a student in *fifth* grade is studying page *100* (the *hundredth* page) of the textbook. Sometimes a number is used in the nominal sense, that is, as a name. A social security number is used this way.

In this guide, the concept of number is extended to include positive rational numbers and then positive and negative integers.

In grades 7 and 8 the emphasis is on the study of rational numbers, negative as well as non-negative. A brief introduction is also included to the irrational numbers which leads the students to the set of real numbers. A more complete development of the set of irrational numbers and the set of real numbers is introduced in grades 9 through 12.

STUDENTS SHOULD BE ABLE TO:

NUMERATION SYSTEMS

Names are given to the cardinal numbers. These names are numerals and are used to convey the idea of number. Numerals are symbolized forms used for communication. A given number has many different names which can be used depending on the idea one wishes to express. Organized methods of arranging the numerals represent the numeration systems of the past and present. Study of different numeration systems will give the student insight and appreciation of the more efficient ones.

The understanding of base and place value of a numeration system is extended in 7th and 8th grades. Expanded notation, scientific notation using positive, zero and negative integral exponents, and the relationship between fractional and decimal representations of rational numbers are emphasized.

STUDENTS SHOULD BE ABLE TO:

ORDER

The concept of order is a generalization made as a result of experience in comparing unmatched sets of objects to determine which has more members. Order is expressed in sentences by means of the symbols > (is greater than) and < (is less than). The concepts of equality and of inequality (numbers that are not equal) are considered here for convenience.

The concept of order is extended from generalizations about cardinal numbers of sets to order in the set of real numbers based on the following two principles.

(1) If a and b are any two real numbers, then one and only one of the following is true:

$a = b$, $a < b$, or $a > b$.

(2) Given three distinct real numbers, one is between the other two.

STUDENTS SHOULD BE ABLE TO:

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

A number system consists of three parts: a set of numbers, two operations defined on the numbers in the set, and the properties (laws or rules) of the operations.

Two number systems, the whole numbers and the positive rational numbers, are studied in detail in the elementary school. A third system, the integers, is introduced in the elementary school when the physical situation demands that the number system be extended to include negative numbers. The integers and a fourth number system, the real numbers, are studied in more detail in junior and senior high school. An understanding of the integer and real number systems and their properties depends on careful and complete development of the whole number and (positive) rational number systems in the elementary school.

The operations on numbers studied in the elementary school are addition and multiplication, and their respective inverses, subtraction and division.

The important and useful properties of the operations on all number systems studied in arithmetic are given here. They are:

Addition

Commutative: $a + b = b + a$
Associative: $(a + b) + c = a + (b + c)$
Identity element: $a + 0 = a$
Inverse element: $a + (-a) = 0$

Multiplication

Commutative: $a \times b = b \times a$
Associative: $(a \times b) \times c = a \times (b \times c)$
Identity element: $a \times 1 = a$
Inverse element: $a \times 1/a = 1$, $a \neq 0$
Property of zero: $a \times 0 = 0$
Distributive: $a \times (b + c) = (a \times b) + (a \times c)$

STUDENTS SHOULD BE ABLE TO:

RATIO AND PROPORTION

A ratio is a pair of numbers used to compare quantities or to express a rate.

Symbols commonly used for a ratio are (a,b), a:b, and a/b. Note that the symbol (a,b) is also used to denote the coordinates of a point in a plane and that the symbol a/b is generally used to name a rational number, when a and b are integers and b is not zero.

A proportion is a statement that two ratios are equivalent (that two pairs of numbers express the same rate). A proportion is written in the form $a/b = c/d$, when $a \times d = c \times b$.

STUDENTS SHOULD BE ABLE TO:

Instructional Objectives-Kindergarten

SETS AND NUMBERS

1. Identify two equivalent sets by placing the members of the set in one-to-one correspondence.
2. Select the set of objects associated with a given number.
3. Count orally by matching numerals with sets having a given number of objects.
4. Identify, without counting, the number of sets with two, three, or four objects.
5. Use such terms as *more than, as many as, fewer than* when comparing sets of objects.

NUMERATION SYSTEMS

1. Identify the numerals 0 through 9.

ORDER

1. Determine whether two sets are equivalent (can be matched or placed in a one-to-one correspondence).
2. Compare two non-matching sets of fewer than 10 objects and decide which set has more members and which set has fewer members.
3. Determine that 3 is greater than 2 and that 2 is less than 3 by comparing appropriate sets of objects and do this for any two numbers less than 6.
4. Utilize the idea "one more than" in organizing sets in the natural order.

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Rearrange sets of objects to demonstrate the joining and separating of sets, and thereby develop a readiness for addition and subtraction.
2. Tell a story about a problem represented by the above activities.
3. Use objects to represent the "action" or conditions of a problem.

RATIO AND PROPORTION

Instructional Objectives-First Grade

SETS AND NUMBERS

1. Count the members of a set containing one hundred or fewer members.
2. Demonstrate one-half, one-fourth of a physical unit.
3. Use the ordinal numbers through fifth.
4. Use "0" as the symbol for the number of elements in the empty set.
5. Insert missing sets, such as a set of 3 elements between a set of 2 elements and a set of 4 elements.

NUMERATION SYSTEMS

1. Read and write any numeral from 0 through 100.
2. Give different numerals for a given number such as 6 + 2, 10 - 2, and 8 for eight.
3. Read number words through *ten*.
4. Interpret the place-value concept for writing whole numbers to one hundred; such as, 89 is the same as 8 tens, 9 ones.

ORDER

1. Compare two non-matching sets of less than 100 objects to decide which set has fewer (more) members.
2. Determine that 8 is greater than 5 and that 5 is less than 8 by comparing appropriate sets of objects and do this for any two numbers less than 10.

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Identify the process of addition through experience with joining two sets of objects.
2. Identify the process of subtraction through experience with separating a subset from a set of objects.
3. Recognize examples of the commutative property for addition in the set of whole numbers.
4. Demonstrate with sets of objects the relationship between such sentences as 4 + 2 = 6, 6 - 2 = 4, and 6 - 4 = 2.
5. Use the symbols +, -, and = to form sentences such as 3 + 6 = □.

RATIO AND PROPORTION

Instructional Objectives-Second Grade

SETS AND NUMBERS

1. Determine the cardinality of a set to one thousand when the objects are already grouped. For example, 4 bundles of 100 sticks, 9 bundles of 10 sticks, and 4 sticks are the same as 494 sticks.
2. Use ordinal numbers through tenth.
3. Identify 1/2, 1/4, 1/3 of a whole by using physical objects.

NUMERATION SYSTEMS

1. Read and write any numeral through 999.
2. Write many symbols for the same number; for example, 6 + 3, 5 + 4, 17 - 8, and 9 for nine.
3. Count by 5s, 10s, and 100s.
4. Use physical objects to demonstrate place value through 999; for example, 234 = 200 + 30 + 4.
5. Write three-digit numerals in expanded notation; for example, 765 = 700 + 60 + 5.
6. Use physical objects to demonstrate regrouping; for example, 2 bundles of 10 sticks and 16 sticks have the same number of sticks as 3 bundles of 10 sticks and 6 sticks. Also 4 bundles of 10 sticks and 8 sticks have the same number of sticks as 3 bundles of 10 sticks and 18 sticks.

ORDER

1. Use the terms *greater than* and *less than*, and *equals* in sentences.
2. Use symbols >, <, and = in mathematical sentences.
3. Insert missing numbers, such as 3 between 2 and 4.
4. Name successors and predecessors of each number through 99.
5. Determine the order of numbers through 99.

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Recognize that there is no largest whole number.
2. Recognize that subtraction is not commutative.
3. Identify the process of multiplication through experience with joining several equivalent sets of objects.
4. Recognize zero as the identity element for addition in the set of whole numbers and its special role in subtraction.

RATIO AND PROPORTION

Instructional Objectives-Third Grade

SETS AND NUMBERS

1. Determine the cardinality of a set to 10,000 through appropriate experiences.
2. Use ordinal numbers beyond tenth.
3. Identify 2/3 and 3/4 of a whole.
4. Show 2/4 = 1/2, etc., by the use of physical objects or pictures.

NUMERATION SYSTEMS

1. Read and write any numeral to 10,000.
2. Write many symbols for the same number, such as 7 + 5, 4 x 3, 10 + 2, and 12 for twelve.
3. Interpret place value to 10,000.
4. Write four-digit numerals in expanded notation; for example, 4567 = 4000 + 500 + 60 + 7.
5. Recognize that numerals such as 57 can be expressed as 40 + 17.

ORDER

1. Determine betweenness, greater than, or less than for numbers through 999.
2. Recognize greater than or less than for the fractions 1/4, 1/3, 1/2 with physical objects. (Note that *fractions* is being used here for *rational numbers*.)

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Recognize the role of 1 as the identity element for multiplication in the set of whole numbers.
2. Recognize the distributive property of multiplication over addition in the set of whole numbers; for example, 3 x (10 + 4) = (3 x 10) + (3 x 4).
3. Demonstrate the concept of division by partitioning a set into several equivalent subsets; for example, separate a set of 12 objects into sets of 3 objects.
4. Demonstrate with sets of objects the relationship between such sentences as 4 x 7 = 28, 28 ÷ 4 = 7, and 28 ÷ 7 = 4.

RATIO AND PROPORTION

1. Interpret simple ratio situations, such as 2 apples for 15¢, written $\frac{2 \text{ (apples)}}{15 \text{ (cents)}}$.
2. Recognize that ratios such as $\frac{2 \text{ (pencils)}}{5 \text{ (cents)}}$ and $\frac{4 \text{ (pencils)}}{10 \text{ (cents)}}$ are equivalent ratios (represent the same rate).

Instructional Objectives-Fourth Grade

SETS AND NUMBERS

1. Determine the factors of a counting number (a whole number other than zero).
2. Determine common factors of two counting numbers.

NUMERATION SYSTEMS

1. Read and write numerals as needed.
2. Interpret place value for large numbers.
3. Round whole numbers to the nearest 10 or 100.
4. Read decimals to tenths.

ORDER

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Recognize the inverse relation between addition sentences and two subtractions sentences, such as 725 + 342 = 1067 and 1067 - 725 = 342 and 1067 - 342 = 725.
2. Recognize the special role of 1 as a divisor.
3. Use parentheses to show order of operation; for example, $2 + 4 \times 3 = 2 + (4 \times 3) = 14$ and $(2 + 4) \times 3 = 6 \times 3 = 18$.

RATIO AND PROPORTION

1. Make up sets of equivalent ratios for given physical situations, such as 1/2, 2/4, 3/6, 4/8, . . .
2. Determine if two ratios are equivalent by using the property of proportions commonly called *cross multiplication*. For example, 3/4 = 9/12 because $3 \times 12 = 4 \times 9$, whereas 6/7 ≠ 7/8 because $6 \times 8 \neq 7 \times 7$.
3. Find the missing whole number in two equivalent ratios like 2/3 = □/9 or 5/□ = 25/70.
4. Use equivalent ratios to convert units of measure, such as $\frac{\text{(gallons) } 1}{\text{(pints) } 8} = \frac{\text{(gallons) } 3}{\text{(pints) } \square}$ to find how many pints there are in 3 gallons.

Instructional Objectives-Fifth Grade

SETS AND NUMBERS

1. Identify prime numbers such as 2, 3, 5, 7, 11, 13, 17, . . .
2. Find the prime factors of numbers through 100.
3. Identify composite numbers.
4. Determine the least common multiple of two counting numbers.
5. Determine the greatest common factor of two counting numbers.
6. Construct sets of equivalent fractions through working with sets of objects. An example of such a set is {2/3, 4/6, 6/9, 8/12. . .}.

NUMERATION SYSTEMS

1. Write many names for the same rational number.
2. Round whole numbers to the nearest 1000.
3. Use simple exponents such as $10^2 = 100$, $3^2 =$ and express 300 as 3×10^2.
4. Read and write simple decimals.

ORDER

1. Determine greater than, less than, and betweenness for rational numbers.

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Recognize that zero is the identity element for addition in the set of positive rational numbers as well as in the set of whole numbers.
2. Recognize that subtraction is not always possible in the set of positive rational numbers and in the set of whole numbers.
3. Recognize that there is no smallest positive rational number.
4. Add, subtract, and multiply simple rational numbers by use of physical objects, diagrams, etc.

RATIO AND PROPORTION

Instructional Objectives-Sixth Grade

SETS AND NUMBERS

1. Use negative numbers in many different situations.

NUMERATION SYSTEMS

1. Represent rational numbers by decimals and fractions.
2. Express large numbers by using scientific notation, such as the distance from earth to the sun as 9.3×10^7 miles.
3. Use exponential notation in representing numbers; for example, $2345 = 2 \times 10^3 + 3 \times 10^2 + 4 \times 10 + 5$.
4. Demonstrate an understanding of the relationship between decimals and common fractions.

ORDER

1. Determine greater than, less than, and betweenness for (positive, negative, and zero) integers.

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Recognize that 1/1 or 1 is an identity element for multiplication in the set of rational numbers.
2. Recognize the multiplicative inverse (reciprocal) for every positive rational number except zero and use it in the division of rational numbers. For example, $1/2 \div 3/4 = 1/2 \times 4/3$.
3. Recognize that the operation of division is the inverse of multiplication in the set of positive rational numbers. For example, the sentences $3/4 \times 2/3 = 1/2$, $1/2 \div 3/4 = 2/3$, and $1/2 \div 2/3 = 3/4$ have this relationship.
4. Recognize that there is no smallest or largest rational number between two positive integers.
5. Find the additive inverse (opposite) for each integer by using the number line.
6. Recognize that finding an integral power of a number involves repeated multiplication of the same number. For example, $(2/3)^3 = 2/3 \times 2/3 \times 2/3$.
7. Use the commutative and associative properties of multiplication for rational numbers.
8. Use the distributive property of multiplication with respect to addition of rational numbers.
9. Use the commutative property of addition for integers.
10. Recognize that the rational number system is dense; that is, between each two different rational numbers, there is a rational number.

RATIO AND PROPORTION

1. Interpret percent as a ratio in which the second number is always 100.
2. Solve all three cases of percentage problems.
3. Use equivalent ratios to convert fractions to decimals and conversely; for example, to write 3/5 as hundredths, solve for n in $3/5 = n/100$; to write 44 hundredths as 25ths, solve for n in $n/25 = 44/100$.
4. Solve ratio problems where some or all of the terms of the ratios are written as decimals.
5. Use proportions in problems about the lengths of sides of similar triangles.

Instructional Objectives-Seventh Grade

SETS AND NUMBERS

1. Given a universal set describe or list the members of a given set. For example, in the universal set of counting numbers the set of multiples of 3 is {3, 6, 9, 12, . . .}.
2. Express a counting number in prime factor form. For example, $24 = 2 \times 2 \times 2 \times 3 = (2^3)\ (3)$.
3. Interpret a fraction as a part of a whole, as expressing a ratio, as part of a group, or as an indicated division.
4. Given a set of numerals, classify them as representing whole numbers, and/or integers, and/or rational numbers.

   Examples: $8/4$ represents a whole number and an integer and a rational number .333. . .represents a rational number; -4 represents an integer and a rational number.

NUMERATION SYSTEMS

1. Use positive integral exponents to express the power of a positive rational number.
2. Recognize that rational numbers can be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.
3. Write any positive rational number in decimal notation.

ORDER

1. Recognize that one and only one of the following statements is true when a and b are any two rational numbers: $a < b$, $a = b$, $a > b$.

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Recognize and apply the addition and multiplication properties of the system of positive rational numbers. (See description of Number Systems, Operations and Their Properties.)
2. Recognize and apply the addition properties of the system of integers.
3. Recognize and apply the distributive property in the system of positive rational numbers.

RATIO AND PROPORTION

1. Recall that a ratio can be used to *compare quantities* as well as to *express a rate*.

Instructional Objectives-Kindergarten

COMPUTATION

SIZE AND SHAPE

1. Recognize squares, rectangles, circles, and triangles.
2. Use the terms *inside*, *outside*, and *on* as related to these figures.

SETS OF POINTS

SYMMETRY

CONGRUENCE

SIMILARITY

CONSTRUCTIONS

COORDINATE SYSTEMS AND GRAPHS

1. Read and construct picture graphs.

Instructional Objectives-Eighth Grade

SETS AND NUMBERS

1. Show that a rational number can be expressed as a repeating decimal, and that a repeating decimal can be expressed in fractional form.
2. Identify non-repeating decimals representing irrational numbers. For example, .232332333233332. . .

NUMERATION SYSTEMS

1. Use positive, zero, and negative exponents correctly.
2. Use scientific notation to express positive rational numbers; for example, the length of a microwave is 2.2 x 10^{-2} meters.
3. Use the symbol $\sqrt{a}$ to indicate one of two equal non-negative factors whose product is a.
4. Use expanded notation to express positive rational numbers.

   Example: $21.37 = 2 \times 10^{1} + 1 \times 10^{0} + 3 \times 10^{-1} + 7 \times 10^{-2}$.

ORDER

1. Determine, given a rational number and an irrational number, which is greater. For example: which is greater 2 or $\sqrt{3}$?

NUMBER SYSTEMS, OPERATIONS AND THEIR PROPERTIES

1. Recognize and apply the addition and multiplication properties of the rational number system.
2. Recognize and apply the multiplication properties of the system of integers.
3. Define subtraction in terms of addition in the systems of integers and rational numbers. For example: a - b means a + (-b).

RATIO AND PROPORTION

1. Use proportions to solve problems involving similar triangles.

Mathematical Concepts

COMPUTATION

Computation can be described as the application of systematic (algorithmic) procedures to the process of renaming numbers.

A certain amount of proficiency in the use of the various algorithms of arithmetic is necessary. The grade placement used for these computational skills in this guide is only an approximation.

STUDENTS SHOULD BE ABLE TO:

SIZE AND SHAPE

The classification of and distinguishing characteristics of two- and three-dimensional geometric figures are determined by their size and shape.

Sufficient time should be allowed for students to examine models, determine their own classification schemes, and to make and test conjectures about properties of plane and solid figures.

STUDENTS SHOULD BE ABLE TO:

SETS OF POINTS

Space can be considered as a set of points. Intuitively, a point represents and is represented by a position or location in space. Lines and planes are subsets of space. Each line is a set of points of space and each plane is a set of points of space. These subsets and many of their properties should be intuitively presented.

STUDENTS SHOULD BE ABLE TO:

SYMMETRY

Many geometric figures and designs have a kind of balance called symmetry. If a figure can be folded so that corresponding parts coincide, then it has a line of symmetry. A figure has a center of symmetry if for every point of the figure there is a second point such that a line segment joining them is bisected by this center.

STUDENTS SHOULD BE ABLE TO:

CONGRUENCE

Intuitively two geometric figures are congruent if they "fit" each other exactly, that is, if they have the same size and shape. More precisely, two sets of points are congruent if there is a one-to-one correspondence between them which preserves distances; that is, if two points of one set are one inch apart, the corresponding points of a congruent set are also one inch apart.

STUDENTS SHOULD BE ABLE TO:

SIMILARITY

Two geometric figures that have the same shape, though not necessarily the same size, are said to be similar. The measures of corresponding dimensions of these figures will, thus, have the same ratio. Ideas of similarity are needed for the interpretation and drawing of building plans, road maps, and scale models.

STUDENTS SHOULD BE ABLE TO:

CONSTRUCTIONS

Students can gain insight to geometry by using a variety of materials and instruments to construct models of geometric figures. Some useful materials and instruments are paper (for folding), wire, string, coins, rulers, compasses, and protractors.

STUDENTS SHOULD BE ABLE TO:

COORDINATE SYSTEMS AND GRAPHS

A line is called a number line if a one-to-one correspondence exists between a given set of numbers and a subset of points on the line and if, by means of this correspondence, the points are kept in the same order as their corresponding numbers. The number corresponding to a point is the coordinate of that point. The idea of assigning numbers to points can be extended to points in a plane, that is, a one-to-one correspondence between ordered pairs of numbers and points in a plane.

STUDENTS SHOULD BE ABLE TO:

Instructional Objectives-First Grade

COMPUTATION

1. Use the addition facts through the sum of 10 and the corresponding subtraction facts.

SIZE AND SHAPE

1. Observe distinguishing features of spheres, rectangular prisms (boxes), cylinders, and other objects.
2. Use the terms *round*, *face*, *edge*, *corner*, and *surface*.

SETS OF POINTS

1. Recognize physical representations of points, line segments, and portions of a plane (flat surfaces).
2. Recognize that squares, rectangles, triangles, and circles are closed curves and tell whether a point is inside, outside, or on such a curve.

SYMMETRY

CONGRUENCE

SIMILARITY

CONSTRUCTIONS

COORDINATE SYSTEMS AND GRAPHS

1. Use the number line to illustrate addition and subtraction problems.

Instructional Objectives-Second Grade

COMPUTATION

1. Recall the addition facts through the sum of 18 and the corresponding subtraction facts.
2. Add three one-digit numbers.
3. Use the multiplication facts through the product 18.

SIZE AND SHAPE

SETS OF POINTS

1. Recognize a point as a position and a line segment or curve as a set of points.
2. Recognize a straight line as a set of points with no beginning and no end.
3. Recognize a simple curve (in a plane) as one that does not cross itself.
4. Recognize closed and open simple curves.
5. Recognize the inside and outside of simple closed curves.

SYMMETRY

CONGRUENCE

1. Recognize congruent, plane figures as figures which fit on one another.
2. Recognize congruent segments as segments having the same length.

SIMILARITY

CONSTRUCTIONS

COORDINATE SYSTEMS AND GRAPHS

1. Use the number line to illustrate counting by fives and by tens.
2. Read and construct bar graphs.

Instructional Objectives-Third Grade

COMPUTATION

1. Use the multiplication facts with products through 45 and the corresponding division facts.
2. Use the vertical algorithm in addition and subtraction with two- and three-place numerals when regrouping may be necessary.
3. Use the vertical algorithm to carry out multiplication with multipliers less than 10 when regrouping may be involved.
4. Multiply mentally by 10 and by 100.
5. Estimate the sum of two numbers. For example, 287 + 520 is approximately 300 + 500 or 800.

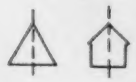
SIZE AND SHAPE

SETS OF POINTS

1. Recognize rays and angles.
2. Recognize that there is only one line through two points and that two lines can intersect at only one point.
3. Recognize that many lines may pass through a point.

SYMMETRY

1. Recognize symmetry with respect to a line by folding paper containing symmetrical figures along their vertical axes of symmetry. For example:



CONGRUENCE

1. Recognize congruent angles.
2. Recognize that a rectangular sheet of paper can be divided into two or more congruent parts through folding.

SIMILARITY

1. Recognize that figures are similar if they have the same shape. For example: all squares are similar.

CONSTRUCTIONS

COORDINATE SYSTEMS AND GRAPHS

1. Recognize that a point on a line can be described by a number (coordinate).
2. Recognize that different (uniform) scales can be applied to the same line.
3. Use the number line to illustrate multiplication problems.

Instructional Objectives-Fourth Grade

COMPUTATION

1. Recall the multiplication facts through 10 x 10.
2. Subtract using three-place numerals.
3. Multiply a number by multiples of 10, by multiples of 100.
4. Use the multiplication algorithm with two-place multipliers.
5. Use the division algorithm with one-place divisors.
6. Estimate the product of two numbers and the quotient of two numbers. For example, 21 x 88 is approximately 20 x 90 or 1800 and 795 ÷ 23 is approximately 800 ÷ 20 or 40.

SIZE AND SHAPE

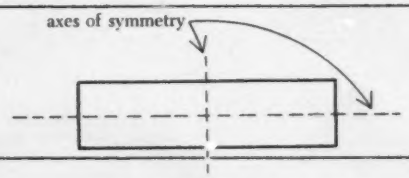
1. Recognize isosceles and equilateral triangles and parallelograms.

SETS OF POINTS

1. Interpret space as the set of all points.
2. Recognize a plane as a flat surface which contains lines and points.
3. Describe lines as intersections of planes.
4. Interpret a circle as the set of all points in a plane that are at the same distance from a fixed point.
5. Recognize parallel lines as lines in a plane which do not intersect.

SYMMETRY

1. Recognize that some figures have two or more axes of symmetry through paper folding. For example, two axes of symmetry are indicated for the rectangle shown at right.



CONGRUENCE

SIMILARITY

1. Recognize that all congruent figures are similar, but not all similar figures are congruent.

CONSTRUCTIONS

1. Demonstrate through paper folding an understanding of a line as an intersection of two planes.
2. Reproduce a line segment by using a compass and straightedge.
3. Bisect a line segment by using a compass and straightedge.

COORDINATE SYSTEMS AND GRAPHS

1. Recognize that a line segment is a set of points.
2. Recognize that points in a plane (the first quadrant) can be represented by (ordered) pairs of numbers (coordinates).
3. Read and construct simple line graphs.

Instructional Objectives-Fifth Grade

COMPUTATION

1. Add and subtract rational numbers.
2. Divide by multiples of 10; by multiples of 100.
3. Express the quotient of integers as a mixed numeral; for example, $24 \div 5 = 4\ 4/5$.
4. Find many ways to express a rational number; for example, $2/3 = \frac{2 \times 4}{3 \times 4} = 8/12$ and $8/12 = \frac{8 \div 4}{12 \div 4} = 2/3$.

SIZE AND SHAPE

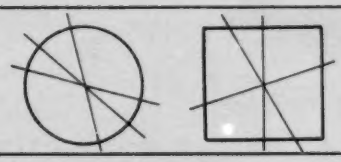
1. Recognize common polyhedra, such as a tetrahedron, a cube, a rectangular prism.
2. Identify faces, edges, vertices, and diagonals of common polyhedra.

SETS OF POINTS

1. Recognize acute, right, and obtuse angles.
2. Recognize parallel planes.
3. Recognize perpendicular lines.
4. Recognize the radius and diameter of a circle.
5. Recognize that a plane is determined by three points not all on one line.

SYMMETRY

1. Recognize symmetry with respect to a point by folding a paper along a line through the center of such geometric figures as a circle and a square.



CONGRUENCE

1. Recognize that triangles are congruent if corresponding sides are congruent and corresponding angles are congruent.

SIMILARITY

1. Recognize the similarity of maps made with different scales.

CONSTRUCTIONS

1. Demonstrate an understanding of various polyhedra by making appropriate paper models.
2. Bisect an angle. (Students may discover several different constructions.)
3. Reconstruct an angle and a triangle by using a compass and a straightedge.

COORDINATE SYSTEMS AND GRAPHS

1. Use the number line to represent positive rational integers.
2. Use the number line to represent negative integers.

Instructional Objectives-Sixth Grade

COMPUTATION

1. Multiply rational numbers.
2. Add integers.

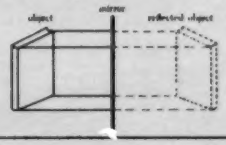
SIZE AND SHAPE

SETS OF POINTS

1. Recognize the properties of isosceles triangles, equilateral triangles, and scalene triangles, such as the fact that the longest side of a triangle is opposite the angle of greatest measure.
2. Recognize that a line (one-dimensional space) is a subset of a plane (two-dimensional space) and that both are subsets of space (three-dimensional space).
3. Recognize the relationship between the circumference and the diameter of a circle.

SYMMETRY

1. Recognize the reflection of a plane figure in a mirror and draw diagrams such as the figure at right.



CONGRUENCE

SIMILARITY

CONSTRUCTIONS

1. Construct a line perpendicular to a given line.
2. Construct parallel lines.
3. Make models of various prisms and find their surface areas.

COORDINATE SYSTEMS AND GRAPHS

Instructional Objectives-Seventh Grade

COMPUTATION

1. Add and subtract integers.
2. Express positive rational numbers in decimal form.
3. Add, subtract, multiply, and divide non-negative rational numbers in decimal form, recognizing that division by zero is undefined (impossible).
4. Average (compute the *mean*) sets of non-negative rational numbers.
5. Compute positive integral powers of non-negative rational numbers. Examples: $3^4 = ?$; $1/2^3 = ?$.
6. Solve all types of percentage problems (including problems involving percents less than one and greater than one hundred).
7. Given any two rational numbers, find a rational number that is *between* them.

SIZE AND SHAPE

1. Classify sets of polygons (quadrilaterals, rectangles, squares, rhombuses, parallelograms, trapezoids, pentagons, hexagons, etc.).
2. Classify angles (right, acute, obtuse); recognize supplementary and complementary angles.
3. Classify prisms and pyramids according to their bases.
4. Identify regular polygons.

SETS OF POINTS

1. Describe the following:

   Half plane
   Angle
   Triangle
   Polygon
   Intersecting lines
   Parallel lines
   Interior regions
   Etc.

SYMMETRY

1. Identify symmetry with respect to a line.

CONGRUENCE

1. Identify corresponding parts of congruent plane figures.
2. Recognize that corresponding parts of congruent plane figures are congruent, ($\cong$).
3. Recognize that circles are congruent if they have congruent radii.
4. Recognize that radii, diameters, corresponding altitudes and corresponding diagonals of congruent plane figures are congruent.
5. Recognize that triangles are congruent if corresponding sides are congruent and corresponding angles are congruent.

SIMILARITY

1. Identify corresponding parts of similar plane figures.
2. Recognize that corresponding angles of similar plane figures are congruent.
3. Recall that measures of corresponding sides of similar plane figures are proportional.

CONSTRUCTIONS

1. Construct a rhombus and a square given one side.
2. Construct an equilateral triangle given one side.
3. Construct the perpendicular bisector of a line segment.
4. Make models of pyramids.

COORDINATE SYSTEMS AND GRAPHS

(See Ordered Pairs, Relations and Functions and Mathematical Sentences)

Instructional Objectives-Eighth Grade

COMPUTATION

1. Add, subtract, multiply, and divide integers and rational numbers.
2. Approximate square roots of positive integers.
3. Express repeating decimals in fractional form.
4. Compute the *mean* of a set of rational numbers.
5. Compute products and quotients of numbers expressed in exponential notation (including scientific notation).

   Examples: $3^7 \cdot 3^5 = 3^{12}$; $a^5 \div a^2 = a^3$; $(3 \times 10^2)(2.3 \times 10^3) = 6.9 \times 10^5$.
6. Perform a series of operations in proper order when grouping symbols are omitted, i.e., multiplications and divisions are performed first, in left-to-right order, then additions and subtractions are performed in left-to-right order.

SIZE AND SHAPE

1. Identify spheres, circular cylinders, and circular cones.
2. Recognize convex and nonconvex polygons.
3. Identify regular solids.

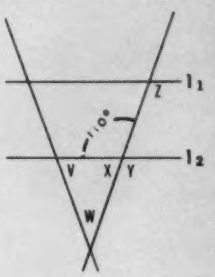
SETS OF POINT

1. Describe the following:

   Skew lines
   Tangent, secant
   Central angle
   Arc
   Diagonal
2. Recognize the various plane figures formed when a plane intersects a given figure such as a square pyramid or a cube.
3. Use relationships such as those involved in finding the solution of the following:

   l_1 is parallel to l_2.
   Find the number of degrees in $\angle x$, $\angle y$, $\angle z$ and the sum of the measures of $\angle v$ and $\angle w$.

   (Relationships involved: vertical angles, supplementary angles, alternate interior angles, exterior angle to remote interior angles of a triangle.



SYMMETRY

1. Identify symmetry with respect to a plane.

CONGRUENCE

1. Recognize that two triangles are congruent if two sides and the included angle of one are congruent to the corresponding parts of the second (SAS).
2. Recognize that two triangles are congruent if two angles and the included side of one are congruent to the corresponding parts of the second (ASA).
3. Recognize that two triangles are congruent if the three sides of one are congruent to the corresponding three sides of the second (SSS).
4. Identify congruent figures in two and three dimensions.

SIMILARITY

1. Recall that the measures of radii, diameters, corresponding altitudes, and corresponding diagonals of similar plane figures are proportional.
2. Recognize that triangles are similar if corresponding angles are congruent.
3. Recognize that triangles are similar if the measures of corresponding sides are proportional.
4. Recall that the ratio of the measures of two sides of a triangle is the same as the ratio of the measures of the corresponding two sides of a similar triangle.
5. Recall that sine, cosine, and tangent ratios are independent of the measures of the sides of the triangles involved.

CONSTRUCTIONS

1. Given three sides, or two sides and the included angle, or two angles and the included side, construct the triangle.
2. Determine whether you can construct none, one, two, or many triangles given three angles or given two sides and an angle not included between them.
3. Construct representations of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, etc.

COORDINATE SYSTEMS AND GRAPHS

(See Ordered Pairs, Relations and Functions and Mathematical Sentences)

Mathematical Concepts

MEASUREMENT

The process of measuring associates a number with a property of an object. Measuring an object is done either directly or indirectly. In direct measurement the number assigned to an object is determined by its direct comparison to a selected unit of measure of the same nature as the object being measured. When a measuring instrument cannot be applied directly to the object to be measured, indirect measurement is employed.

Measure of physical objects is approximate. The accuracy of the measure obtained is restricted by the unevenness of the object measured, by the limitations of the measuring instrument used, and by human inabilities. The object to be measured must be measured by any unit with the same characteristics; a unit segment to measure segments; a unit angle to measure angles; a unit closed region to measure closed regions; and a unit solid to measure solids. Students should learn to estimate measures and use measuring instruments calibrated in metric as well as customary (English) units.

STUDENTS SHOULD BE ABLE TO:

MATHEMATICAL SENTENCES

Mathematical sentences and ordinary linguistic sentences have the following common characteristics:

(1) Both types of sentences use symbols to communicate ideas.
(2) The construction of both types of sentences follows a predetermined set of rules.
(3) Both types of sentences may express true statements or false statements depending upon the symbols used and the contexts in which they are used.

It is quite common in mathematics, however, to use sentences that do not possess characteristic 3. Such sentences are called *open* sentences. For example, the sentence $\square + 3 = 7$ expresses neither a true statement nor a false statement until a meaningful replacement for $\square$ has been supplied. The set of all replacements for $\square$ which produce a true statement is called the *solution set* for the given sentence. Consider the sentence $12 - N > 5$. The set of all allowable replacements for N is sometimes called the universe. Thus, if the universe is the set of positive integers, the solution set is {1,2,3,4,5,6}; but, if the universe is the set of rational numbers, the solution set consists of all rational numbers less than 7, an infinite set.

STUDENTS SHOULD BE ABLE TO:

ORDERED PAIRS, RELATIONS AND FUNCTIONS

An ordered pair of numbers is simply a pair of numbers such as (4,7). The order in which the numbers are named in the pair is important. Thus the ordered pair (4,7) is not the same as the ordered pair (7,4).

Ordered pairs may be used to represent rates, comparisons, integers, rational numbers, vectors, and coordinates of a point in a plane.

A *relation* is a set of ordered pairs. A *function* is a relation where the first element in each ordered pair is unique (not the same as any other first element).

Students should become familiar with the basic properties of ordered pairs, relations, and functions and should know how to represent relations and functions graphically. These ideas (ordered pair, relation, function) should be introduced using an intuitive approach.

STUDENTS SHOULD BE ABLE TO:

STATISTICS AND PROBABILITY

Statistics refers to the ways of collecting, organizing, analyzing, interpreting and summarizing numerical data of all kinds. Inferential statistics refers to a means of using data from a relatively small representative sample to predict information about a total population. Since sample data does not give exact information about the population, the art of decision-making or inference from incomplete data involves some knowledge of the theory of probability. The level of probability that one assigns to an inference about a certain population indicates the degree of certainty with which one can expect the same result to occur if representative samples of the same size are repeatedly drawn from the same parent population.

The study of probability includes the study of experiments involving chance events, the outcomes of such experiments, and the likelihoods that particular outcomes will occur. The *probability* of an outcome for a particular experiment is a numerical measure of the likelihood that the outcome will occur.

STUDENTS SHOULD BE ABLE TO:

COMPUTERS

An essential outcome of contemporary education is computer literacy. Each student should acquire an understanding of the versatility and limitations of the computer through firsthand experiences with applications in a variety of subject matter fields. Programs designed to take full advantage of the multidisciplinary potential of computers should include:

- problem solving
- simulations that give the opportunity to practice decision making
- tutorial lessons that introduce and develop concepts
- word processing to develop communication skills
- data retrieval tasks such as career information services
- simulations that replace dangerous, expensive, or technically difficult laboratory work
- programming
- drill-and-practice programs
- impact, use, misuse, and limitations of computers

STUDENTS SHOULD BE ABLE TO:

Instructional Objectives-Kindergarten

MEASUREMENT

1. Recognize pennies, nickels, dimes.
2. Make comparisons in time and count whole units of time (day, week, month, year).
3. Use appropriately such words as *longer, shorter, heavier, lighter, higher, lower, larger, smaller.*

MATHEMATICAL SENTENCES

ORDERED PAIRS, RELATIONS AND FUNCTIONS

STATISTICS AND PROBABILITY

COMPUTERS

Instructional Objectives-First Grade

MEASUREMENT

1. Determine which of two line segments is the longer or the shorter, or whether they are the same length.
2. Recognize the comparative value of coins (pennies, nickels, dimes) and use them in making change.
3. Tell time to the nearest half-hour.
4. Identify various instruments of measurement of time, temperature, weight, and length, such as clocks, thermometers, scales, rulers.
5. Use non-standard units of linear measure and liquid measure, such as a pencil or book for length, and a paper cup for liquid measure.

MATHEMATICAL SENTENCES

1. Write an appropriate mathematical sentence like $3 + 4 = \square$ or $5 - 2 = \square$ for a physical situation or a story problem where the "action" of the problem suggests the operation of addition or subtraction.
2. Find the "solution" for sentences like $3 + 4 = \square$ and $5 - 2 = \square$.
3. Find solutions for sentences like $\square + \bigcirc = 7$ in which many correct solutions are possible.
4. Make up a problem situation to fit a given mathematical sentence involving addition or subtraction.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

STATISTICS AND PROBABILITY

COMPUTERS

Instructional Objectives-Second Grade

MEASUREMENT

1. Make a ruler with divisions showing half units.
2. Use standard units to the nearest whole unit for linear measure, for weight measure, and for liquid measure.
3. Identify proper instruments for measuring different objects.
4. Tell time to the nearest quarter hour.
5. Make change correctly for quantities up to 25¢.

MATHEMATICAL SENTENCES

1. Use sentences like $5 + \square = 12$, $\square + 6 = 8$, $12 - \square = 8$, and $\square - 5 = 6$ to represent physical situations and find solutions for the sentences.
2. Make up story problems to fit sentences like those shown above.
3. Find solutions for sentences like $3 + 2 = 8 - \square$, $\square + 5 = 8 + 7$, $8 + \square < 12$, and $4 + 9 > \square + 5$, with the aid of sets of objects or the number line.
4. Place the correct symbol (+ or -) in the placeholder in sentences like $13 \bigcirc 5 = 8$ and $4 \square 3 < 5$.
5. Use sentences like $27 = 20 + 7$ and $43 = 30 + 13$ to indicate regrouping or the use of different symbols for the same number.
6. Use equivalent sentences like $3 + \square = 7$ and $7 - \square = 3$ to show subtraction as the inverse of addition.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

STATISTICS AND PROBABILITY

COMPUTERS

1. Locate and run pre-existing programs.

Instructional Objectives-Third Grade

MEASUREMENT

1. Use common standard units in determining the measure of a distance.
2. Use standard units of measure in determining capacity and weight.
3. Find the perimeter of a rectangle or parallelogram.
4. Make change for any purchase up to $1.00.

MATHEMATICAL SENTENCES

1. Use sentences like 3 x 5 = □, □ x 7 = 14, and 4 x □ = 12 to represent physical situations and find solutions for the sentences.
2. Make up story problems to fit sentences like those shown above.
3. Place the correct symbol (<, >, =) in the placeholder in sentences such as 3 x 5 □ 7 + 8, 25 + 42 □ 87 - 28, and 65 - 39 □ 5 x 7.
4. Demonstrate understanding of grouping and regrouping by completing sentences such as 458 = □ + 50 + 8 and 394 = 3 hundreds + □ tens + 4 ones, by means of tally boxes or other devices.
5. Find solutions for sentences like □ + 239 = 239 + □ and 1987 + (□ + 548) = (1987 + □) + 548 to generalize the idea of the commutative and associative properties for addition.
6. Use many different kinds of placeholders, like □, ○, N, X in mathematical sentences.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

STATISTICS AND PROBABILITY

1. Prepare a simple tally chart and draw its graph.

COMPUTERS

1. Recognize specific uses of computers in a variety of fields.
2. Identify a variety of computer related careers.
3. Use computerized systems for tutorials, drill-and-practice, and simulations.

Instructional Objectives-Fourth Grade

MEASUREMENT

1. Compare measures such as: 25 inches and 2 feet; 31 ounces and 2 pounds; 75 seconds and 1 minute; and 750 milliliters and 1 liter.
2. Express different names for the same measure.
3. Measure perimeters of triangles and quadrilaterals.
4. Find areas of simple regions informally. For example, a rectangular region with dimensions 2 cm by 3 cm can be covered by six one-centimeter squares (regions).

MATHEMATICAL SENTENCES

1. Use sentences like $36 \div 4 = \square$ and $\square \times 3 = 12$ to represent physical situations and find solutions for the sentences.
2. Make up story problems to fit sentences like those shown above.
3. Use sentences like $\square \times 5 = 45$ and $45 \div 5 = \square$ to show division as the inverse of multiplication.
4. Find solutions for sentences like $723 \times \square = \square \times 723$ and $(\square \times 176) \times 19 = \square \times (176 \times 19)$ to generalize the idea of the commutative and associative properties of multiplication.
5. Find solutions for sentences like $3 \times 13 = (\square \times 10) + (\square \times 3)$, $\square \times \quad = (5 \times 7) + (5 \times 8)$, and $\square \times 16 = (\square \times 10) + (\square \times 6)$ to generalize the distributive property of multiplication over addition.
6. Find solutions for sentences like $\square \times \quad = 36$.
7. Recognize that $3 \times \square = 7$ has no whole number solution.
8. Find solutions for mathematical sentences involving more than one operation such as $(2 \times 5) + 4 = \square$ and $(3 \times 2) + \square = 10$.
9. Make up problem situations to fit mathematical sentences involving more than one operation. For example, make up a story to fit the sentence $(3 \times 4) + 2 = \square$.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

STATISTICS AND PROBABILITY

1. Construct line graphs and circle graphs. Determine the mean for a set of data. Read maps, schedules and time-tables. Find the mean, median, mode and range for a set of data.

COMPUTERS

1. Type and run a pre-existing program.
2. Give the correct output for a simple algorithm.
3. Identify institutions which depend on computers.

Instructional Objectives-Fifth Grade

MEASUREMENT

1. Measure an angle by using a protractor.
2. Demonstrate that the sum of the measures of the angles of a triangle is 180° by tearing off and matching corners of a triangular piece of paper.
3. Find the area of a plane region, such as a rectangular region.
4. Recognize informal concepts of volume. For example, a box with dimensions 2" by 3" by 4" contains 24 one-inch cubes.
5. Recognize that a right angle has the measure 90°.
6. Estimate distances to the nearest unit.
7. Recognize that all measurement involves approximation.

MATHEMATICAL SENTENCES

1. Use all of the previously introduced sentence forms with fractions.
2. Write sentences using fractions to represent physical situations.
3. Use previously described sentence forms to generalize the commutative property of addition, the associative property of addition, and the distributive property of multiplication over addition for rational numbers in fractional form.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

STATISTICS AND PROBABILITY

1. Read and interpret the information about a set of data presented in the form of a frequency table or a graph.
2. List and/or count all the possible outcomes for an experiment for which the simple events are single elements. For example, spinning a spinner, tossing a die, drawing a lot, flipping a coin.
3. Assign probabilities to the outcomes of an experiment for which the simple events are equally likely. For example, the probability of tossing a "2" with one toss of a die is 1 chance out of 6 or 1/6.

COMPUTERS

1. Explain the purpose of a given simple algorithm.
2. Identify the major components of a computer system.
3. Identify factors which limit computer use in specific applications.

Instructional Objectives-Sixth Grade

MEASUREMENT

1. Find the volume of a rectangular prism.
2. Estimate and compare perimeters of polygons, such as rectangles, triangles, and parallelograms.
3. Estimate the area of an irregular plane region by use of a grid where an approximation to the area is the average of the inner and outer areas.
4. Use formulas for the areas of rectangles, parallelograms, and triangles.
5. Use the formula for the circumference of a circle.
6. Use formulas of volume for common solids.
7. Work with approximate numbers. For example, know that the area of a square whose sides measure 6.5 and 3.6 inches to the nearest tenth of an inch has an area between 6.4 x 3.5 and 6.6 x 3.7 square inches.
8. Solve problems involving the measurement of inaccessible heights and distances indirectly by using the properties of similar triangles.

MATHEMATICAL SENTENCES

1. Use all of the previously introduced sentence forms with decimal numerals.
2. Write sentences using decimal numerals to represent physical situations.
3. Use previously described sentence forms to generalize the commutative property of multiplication, the associative property of multiplication, and the distributive property of multiplication over addition for rational numbers in any form.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

1. Find the image of a element under a function described by a table, graph, or equation.
2. Find the pre-image of an element under a function described by a table, graph, or equation.
3. Graph simple functions.

STATISTICS AND PROBABILITY

1. Organize and present data using a frequency table and graphs.
2. Recognize when the outcomes for a given experiment are equally likely. For example, when flipping a fair coin the outcomes "head" and "tail" are equally likely.
3. Assign probabilities to the outcomes of experiments for which the simple events are not equally likely. For example, the probability of the arrow stopping on red on the spinner at the right is 1 chance out of 4 or 1/4.

COMPUTERS

1. Modify a simple algorithm to accomplish a new but related task.
2. Identify advantages and disadvantages of a data base containing personal information on a large number of people.

Instructional Objectives-Seventh Grade

MEASUREMENT

1. Use various measuring instruments (such as rulers having English and Metric scales, protracters, calipers) and record results to the nearer smallest unit of the scales. Example: Use rulers calibrated in tenths of an inch, in sixteenths of an inch, in centimeters and millimeters, etc.
2. Determine the perimeters and areas of regular polygons.
3. Use the formula for the area of a circle.
4. Determine the surface area and volume of prisms and pyramids.
5. Recall that the sum of the measures of the angles of a triangle is one hundred eighty degrees.
6. Determine the sum of the measures of the angles of a quadrilateral.
7. Determine greatest possible error for any calibration of a given measuring instrument.
8. Recognize the relationship between *precision* and *greatest possible error*.
9. Estimate linear measurements without using measuring instruments. Example: The length of a building, the width of a street, the height of a flagpole.
10. Determine the areas of figures such as those below:

MATHEMATICAL SENTENCES

1. Establish the truth value of simple mathematical sentences. Examples: $3 + 4 = 8$ is *false*; $35 \times 24 < 30 \times 20$ is *false*; $13 - 7 \neq 5$ is *true*.
2. Find the solution sets of simple open sentences using as a universe the whole numbers. Example: $3N + 4 = 16$.
3. Graph using the number line solution sets for equalities and inequalities where the replacement set is the set of integers, such as $X > -2$; $X + 3 < 12$; $X3$; $X > -3$ and $X < 5$; $X > 5$ and $X < 4$; $X > -2$ or $X < 3$; $X < 5$ or $X < 3$; $X > 4$ or $X < -2$.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

1. Determine equality of two given ordered pairs of numbers. Example: $(2,3) \neq (3,2)$.
2. Use ordered pairs of real numbers as coordinates of points in a plane. Example: $(2,3)$, $(-5,7)$, $(-1/2,-3)$, $(\sqrt{2},-8)$.
3. Identify *relations* such as parallelism, perpendicularity, equality, not-equal-to, congruence and similarity.
4. Identify *relations* such as is-a-factor-of, less than, greater than, is-a-subset-of, is-an-element-of, is-the-square-of, etc., and formulate a set of ordered pairs of elements from each. Example: Using the relation is-a-factor-of in the following set of numbers, {1,2,3,6} formulate the set of ordered pairs identifying the relation. Answer: {(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)}.
5. Graph relations (including functions) in the planes I x I (integers), Q x Q (rationals), and R x R (reals).

STATISTICS AND PROBABILITY

1. Distinguish between certain and uncertain events.
2. Recognize that the probability of an event is a number p such that $o \leq p \leq 1$, and recognize that the probability of a certain event is 1 and the probability of an impossible event is 0.
3. Recognize that the probability of an event does not guarantee how often the event will occur. For example, the probability of obtaining a "2" when a fair die is tossed is 1/6; however, this does not mean that the "2" will occur once out of *every* six tosses.
4. Estimate the probability of an outcome of an experiment by empirical methods. For example, estimate the probability of a bottle cap landing with the cork side up when it is flipped (the student can flip a bottle cap 100 or more times, keeping a record of how the cap lands, and estimate the probability from the data.
5. List and/or count all the possible outcomes for an experiment for which the simple events are ordered pairs, simple combinations, or simple permutations. For example, flipping two coins, tossing a pair of dice, rearranging the order of three letters of the alphabet, spinning a spinner and simultaneously tossing a coin.

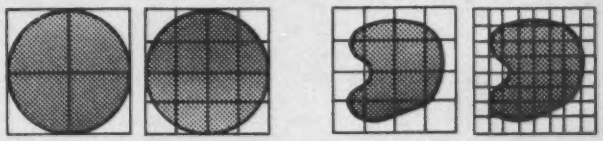
COMPUTERS

1. Develop an algorithm for solving a specific problem.
2. Interpret and use data from computer output to make decisions or solve problems.

Instructional Objectives-Eighth Grade

MEASUREMENT

1. Use formulas to calculate volumes and surface areas of spheres, cylinders, and cones.
2. Determine the relative error given a measuring instrument and an object to measure.
3. Determine the sum of the measures of the angles of regular polygons.
4. Recall the relationship between degrees, minutes, and seconds as units of angular measurement.
5. Recall the relationships between units of the Metric System.
6. Do computation with measures (approximate numbers) and determine the greatest possible error and the relative error of each result (addition, subtraction, multiplication, division).
7. Use the Pythagorean relationship to determine the length of any side of a right triangle given the lengths of the other two sides.
8. Solve problems involving indirect measurement using the sine, cosine, and tangent ratios (including use of tables).
9. Improve results obtained by estimating the area of plane figures through the use of grids by using grids containing smaller units. Examples:



MATHEMATICAL SENTENCES

1. Find solution sets of simple open sentences using as a universe the integers. Examples: $5N + 4 = -16$.
2. Graph using the Cartesian plane I x I (integers) solution sets of inequalities and equalities such as $x + y \neq 19$; $x + y > 19$; $x + y < 19$; $x + y = 19$.
3. Define equivalent open sentences as sentences having the same solution set.
4. Find solution sets of open sentences by finding equivalent open sentences using the properties of equality.

ORDERED PAIRS, RELATIONS AND FUNCTIONS

1. Given a domain (such as a subset of the integers) and given a description of a relation, determine the set of ordered pairs and graph the relation. Example: Domain is {-3,-2, -1,0,1,2,3} and the description of the relation is "double the number and add 3." The determined set of ordered pairs should be {(-3,-3), (-2,-1), (-1,1), (0,3), (1,5), (2,7), (3,9)}.
2. Determine whether a given relation is or is not a function.
3. Given the graph of a linear function, determine a set of ordered pairs and describe the function.

STATISTICS AND PROBABILITY

1. Assign probabilities to the outcomes of an experiment for which the simple events are ordered pairs, combinations, or permutations. For example, the probability of getting 2 heads when 2 coins are flipped simultaneously is 1 out of 4 or 1/4.
2. Recognize how the probability of a specific outcome will change from trial to trial when sampling is done without replacement. For example, from an urn containing 3 red marbles and 2 white marbles the probability of drawing a red marble is 3/5, but the probability of drawing a red marble if the first is not replaced then is either 2/4 or 3/4.
3. Estimate the probability of an event from previous data. For example, a basketball player made 50 free throws out of 75 tries; what is the probability that he will make his next free throw attempt?
4. Estimate information about a population by random sampling. For example, out of 100 flash bulbs selected at random 5 were defective. How many out of 3000 are likely to be defective.

COMPUTERS

1. Write programs that assign data to memory.
2. Develop an algorithm that can be used to solve a set of similar problems.

Problem Solving

A situation becomes a problem for a student only if the following conditions exist:

1. A question is posed for which an answer is not immediately available.
2. The student wants to find a solution.
3. The student has enough confidence to believe a solution is possible.
4. The student is required to use more than immediate recall or previously established patterns of action to find a method for arriving at a conclusion.

Not all pupils will consider a given situation a problem. If a student can see a "method of solution" immediately, then the situation is not a problem for him, but simply an exercise or an application of some process which he has already mastered. For example, an exercise such as 3×14 is not a problem for a student who has already mastered the idea of multiplication with two place numerals. However, this example could very well be a problem for a student who has studied only a few elementary multiplication combinations. In order for this student to obtain a correct solution, some method of renaming 14 such as $10 + 4$ and find some method of reducing the problem to simpler stages, such as $3 \times 14 = 3 \times (10 + 4) = (3 \times 10) + (3 \times 4)$, must be found.

Real problem solving is not recalling a numerical fact or fitting a situation into a memorized pattern, but is discovering one's own method for extending previous learnings to new situations.

It is not the intent of this chapter to consider all of the general aspects of problem solving. Only some of the problem situations that involve the use of known mathematical ideas by students will be discussed. The verbal problems found in textbooks make up a large portion of the problem situations considered in mathematics programs. When such problems are presented, the mathematical ideas necessary to solve them are generally known to the student, but the contexts in which they are presented require the student to find the methods of solution or to decide which of the ideas already known will help in finding the solution.

The solution of a typical textbook verbal problem involves a series of four interrelated steps. The complexity of each of the steps varies with the problem, but the method of solution follows this general pattern.

Step 1—Recognizing the question being asked.
Step 2—Translating the verbal problem into a mathematical sentence.
Step 3—Finding a solution for the mathematical sentence.
Step 4—Analyzing the solution for the sentence to see if it provides a reasonable solution for the original verbal problem.

Each step of the solution can involve many facets. Although all steps in the solution of a verbal problem are important, Steps 1 and 2, the abstraction or translation phase, can be considered the "heart" of the process.

Many decisions may have to be made before the mathematical sentence which represents the problem can be written. If the problem is not clearly stated, the student may need to reformulate it as a verbal problem. Is the data necessary for a solution available? Is some data given which is not necessary for a solution? Which mathematical operation is suggested by the "action" of the problem? In what order will the data appear in the corresponding sentence? What previously solved problems might serve as "models?" If more than one strategy will work, which is easiest?

Steps 1 and 2 could also involve some trial and error activity in which different sentences are tested as to which one best fits the given situation. These steps include problem solving techniques commonly referred to in many textbooks and teacher guides as "Reading and understanding the problem" and "Restating the problem in your own words."

Step 3 involves finding a solution to the mathematical sentence. This procedure can be very simple or very complex depending on the given situation. In general this step involves the application of knowledge that the student has already attained through practice work with similar sentence forms.

Step 4 is the familiar "check" in which the student determines if the solution is acceptable for the given problem situation. The student must decide if the solution is reasonable or "makes sense." If it is apparent that the solution is not correct, the student must then "rework" the problem, check the computations, examine the sentence to see if it actually symbolizes the "story" of the problem, and look for careless errors or possible misinterpretations or misrepresentations of the data.

Some examples of typical verbal problems are given below.

In the first grade, simple "picture" or oral problems are presented which can be represented by sentences like $5 + 2 = \square$ where the sum does not exceed ten. The problem might be presented to the students in this form: "John has 5 toy cars. His mother gave him two more toy cars for his birthday. How many toy cars does he have now?" The "action" of the story is the joining of two sets of toy cars, and the mathematical sentence which represents this problem is $5 + 2 = \square$. Through such presentations, the student will learn to associate the notion of "joining" with the operation of addition. This promotes a better understanding of what addition means and develops techniques for solving similar problems.

In the second and third grades, problem situations are extended to include verbal problems represented by sentences like $3 + \square = 8$, $\square + 5 = 9$, $7 - 2 = \square$, $8 - \square = 5$, and $\square - 2 = 6$. A typical verbal problem at these grade levels is: "Mary had 15 records. After she gave some of the records to her brother, she found that she had 8 records left. How many records did she give to her brother?" This problem is represented by the sentence $15 - \square = 8$. The "action" of the problem is "taking away;" therefore, the corresponding operation of the sentence must be subtraction. The 15 represents the number of records Mary had in the beginning, the $\square$ represents the number of records she gave to her brother, and the 8 represents the number of records she had left. Students should realize that although $15 - \square = 8$, $8 + \square = 15$, and $15 - 8 = \square$ can be represented by different physical situations, the computation involved in solving each of the sentences is the same.

In the later elementary grades, problems are presented that involve larger numbers as in sentences such as $48 + \square = 327$ or $239 - \square = 76$ and that use the same principles as do the problems first introduced in the primary grades.

In the third and fourth grades, problem situations represented by sentences such as $3 \times 7 = \square$, $5 \times \square = 20$, and $\square \times 6 = 30$ are introduced. A sample problem for this grade level is: "Ann found that she needed sixteen small cartons to cover the bottom of a packing case. If she needed four layers of cartons to fill the case, how many small cartons were in the case?" The "action" of the problem is repeated addition or multiplication. Thus the sentence used to represent the story could be $16 + 16 + 16 + 16 = \square$ or $4 \times 16 = \square$. The student should be allowed to use either sentence to represent the problem, but should be led to realize that the sentence $4 \times 16 = \square$ is the shorter way of writing a sentence for this type of problem. In the latter sentence, the 4 represents the number of sets, the 16 represents the number of objects in each set, and the $\square$ represents the total number of objects in all of the sets.

In the sentence 5 x □ = 30, the 5 represents the number of sets, the □ represents the number of objects in each set, and the 30, the number of objects in all. Likewise the sentence □ x 6 = 30 represents a problem situation in which the total number of objects (30) is given, the number of objects in each set (6) is given, and the number of sets, represented by the □, is the solution to the problem. The sentences 5 x □ = 30 and □ x 6 = 30 represent different physical situations, but both types of sentences can be solved by the same computational method of repeated subtraction if the multiplication fact that makes the sentence true is not known.

Problem situations introduced at the fourth and fifth grade levels are similar to the problem: "If John separates 12 marbles into small groups with 4 marbles in each group, how many groups of marbles will he have?" This problem can be represented by the sentence □ x 4 = 12, for the question of the problem can be restated: "How many sets of 4 are there in 12?" The problem can also be interpreted as the division of a set of 12 objects into sets of 4 objects. With the latter interpretation, the problem can be represented by the division sentence 12 ÷ 4 = □. Both sentences, □ x 4 = 12 and 12 ÷ 4 = □, represent the same physical situation and implicitly ask the same question, "How many sets of 4 are there in 12?" Both sentences are solved by repeated subtraction if the number which makes the sentences true is not known.

As fractions, decimal numerals, and integers are introduced, problem situations involving their use in sentence forms similar to those considered previously are presented. It's important that students be able to write the sentence representing the "story" of a verbal problem.

The verbal "story" type problems are not the only source of problem situations for use in the school mathematics program. Many other mathematical problems arise from physical situations, from social or mathematical applications of mathematical ideas, or from situations made up by the teacher or the students. These varied situations can be real, imagined, or in the nature of a puzzle.

Many of these problems fall into a category which might be regarded as a "higher order" of problem solving than the typical textbook "story" problem. It is not possible to outline a regular progression of steps to be followed in solving all such problem situations as the process will vary with the nature of the problem presented and the ability and experience of the student. The four steps outlined for the solution of verbal problems do not necessarily apply in general problem solving situations, for it is not always possible to translate such a problem into a mathematical sentence.

However, it is possible to list a few of the activities that are essential in the general problem solving process. These activities include: translating the problem into a simpler form, critically examining the given data, forming hypotheses or conjectures, reasoning on a trial and error basis, analyzing or evaluating results on the basis of past experience, and forming generalizations from similar problem situations. These activities are not the only ones that may be involved, and not every problem situation will require the student to become engaged in all of the activities listed above. It is important to note that the order in which the student performs these activities may vary for different problem situations.

A few examples of "higher order" problems that can be included in the elementary mathematics program are given below. The types of problems presented at various grade levels should vary only in the complexity of the thinking required, not in the method of thinking or the kinds of mental activities necessary to solve the problems.

Example 1—I have 80 cents in coins in my pocket. I have the same number of pennies, nickels, and dimes. How many coins of each kind do I have in my pocket?

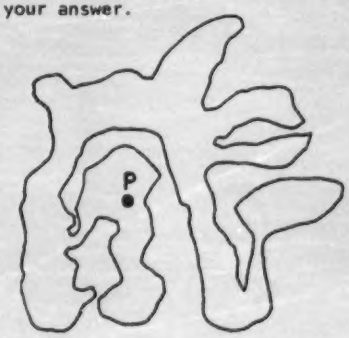
Answer —Five coins of each kind.

Example 2—Fill in the boxes so that the numbers in each column, each row, and each diagonal have a sum of 18. (Do not use any number more than once.)

□	□	7
11	□	□
□	3	□

Answer — 2 9 7
11 6 1 Other answers are possible.
5 3 10

Example 3—Is the point P inside or outside the closed curve? Tell the class the method you used to find your answer.



Answer —The point P is inside the closed curve. If we place another point outside the curve, we *must* cross the curve in order to draw a continuous path between the two points. Also note that any straight line connecting P to an outside point cuts the figure an odd number of times, if you count a point of tangency as two times.

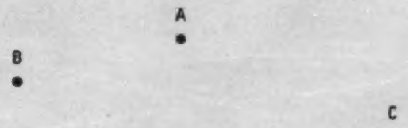
Example 4—Draw two straight lines through the clock face, dividing it into three parts, so that the number represented in each of the three parts have the same sum.

Answer —

Example 5—If 2, 3, 5, and 7 are examples of prime numbers, is 103 then a prime number?

Answer —103 is a prime number as it can be divided evenly only by 1 and itself. (1 and 103 are the only whole number factors of 103.)

Example 6—Points A, B, and C all lie on the same circle. Find the center of the circle that contains points A, B, and C.



Answer —Construct the perpendicular bisectors of two of the segments joining the three given points. The point where the perpendicular bisectors meet is the center.

Example 7—What number can be divided by 11 so that the remainder is zero, but when divided by 2, 3, 4, 5, or 6 leaves a remainder of 1?

Answer —121.

Example 8—Roy had four cards with 1/2 written on the first card and three other numerals written on the other three cards. Roy claimed that he could form all of the combinations from 1/2 to 20, counting by 1/2 (that is, 1/2, 1, 1 1/2, and so on), by using a single card or by adding or subtracting the numbers represented on two or more of the cards. What were the numerals Roy had written on the other three cards?

Answer —1 1/2, 4 1/2, 13 1/2.

The teacher should provide some problem situations similar to the examples above in addition to the typical "story" type problems.

Teachers can do much to help students improve their problem solving abilities for all types of mathematical problems. The teacher's function should be to create a questioning, challenging atmosphere; to introduce problem situations; and to guide and encourage students to develop their own problem solving techniques.

A few suggestions that can help the teacher promote good problem solving techniques are listed below.

- Present problem situations that involve many basic mathematical principles. Be certain that these situations are related to the kinds of experience the pupil has already had so that the pupil is able to apply the principles.
- Let students use their own methods. Many problems have no single, best method of solution. Insistence on one "favorite" method of solution often destroys enthusiasm and original thinking.
- Whenever the problem permits, emphasize the writing of a mathematical sentence that shows the "action" of the problem. If children are to acquire good problem solving habits, they must be able to describe the action of the problem in terms of mathematical symbols whenever applicable. Writing the mathematical sentence for a problem is as important mathematically as finding the solution for the sentence.
- Encourage students to use diagrams, estimates, dramatizations, or other techniques that help them understand the problem.
- Suggest that students try various approaches to a problem when they are not certain of the correct method. Students can evaluate the method used by ascertaining that the solution is reasonable. Students should realize that the trial and error method can be an effective approach to difficult problems.
- Confront students with some situations in which they must formulate their own questions. This type of presentation is similar to the kinds of problems they are apt to face in their future vocations.

For a more detailed treatment of problem solving, see the Department of Public Instruction publication, Problem Solving . . . a basic mathematics goal, volume 1, Becoming a better problem solver, and volume 2, A resource for problem solving.

Adjusting Mathematics Programs to Student Abilities

PRELIMINARY CONSIDERATIONS

The fact that individual differences do exist among students is not only one of the most consistently documented findings of educational research, but also one characteristic of any group of children obvious to every teacher, experienced or novice. Furthermore, this classroom reality fairly demands that educators do something about it. Teachers feel great pressure to adjust the program of instruction so it corresponds to the varying abilities of their students. This fact is especially true in mathematics instruction where the range of abilities encountered is extraordinarily large. Unfortunately, the techniques for gearing programs to individual differences are not as obvious or readily available as is the need for it. It is fair to say that the problem of individualizing instruction is one of the major unsolved problems in contemporary education. In order to understand this situation, it is necessary to examine the kinds of individual differences that one must take into account in attempting to solve this problem.

Differences in levels of mathematical achievement, mental age, attitudes, beliefs, maturity, motivation, talent, and vocational aspirations, exist among children. These differences have a definite impact on the effectiveness of instruction. Although each of these factors may be an important consideration in the construction of instructional programs, they are nevertheless variables which represent only one dimension of the problem. Differences in schools, in school systems, in curricula, and in teachers must be taken into consideration in planning any serious program of individualized instruction.

Teachers, as a group, differ in probably as many ways as do their pupils. From the standpoint of mathematics instruction, the main differences in teachers that need to be considered are those of subject matter interest and knowledge. In mathematics instruction, the teacher's attitude toward the subject may be as influential in determining effectiveness as subject matter knowledge. In any case, both are important, and no effective program of mathematics instruction is possible with serious deficiencies of this type.

Individual differences in schools and curricula form yet another dimension of the problem. Schools and school systems are as different as the people who comprise them. There are inner city schools and suburban schools, each with distinct problems. The curriculum and philosophy of education of these schools also vary greatly. Some systems have multi-tracked curricula; others have single-tracked. Although most schools establish a common pace and instructional content for all pupils, some schools experiment with individualized pacing and even content selection. Some schools tend to rely on the self-contained classroom taught by one teacher. Others use semi-departmentalized concepts of organization. Non-graded and team teaching patterns of organization are not uncommon. Utilization of such devices as closed circuit TV, educational network TV, programmed instruction, and computer-assisted instruction is evident. Although the influence and effects of these variables are often subtle and difficult to analyze, they are real and, in fact, may quite possibly become the controlling factors in efforts to individualize mathematics instruction.

CLASSROOM MANAGEMENT AND GROUPING DEVICES

A primary obstacle to individualized instruction is related to problems of classroom management. It is easy to lecture the class en masse, to test them, to sense who is competent and who is not, and so on. But to individualize instruction, the teacher must try to keep track of what each pupil in the class is doing. With different pupils doing different work, the conventional base of comparison is lost, and evaluation of the pupils' progress becomes increasingly complicated. Finally, the demands of a classroom of individuals for individual attention can easily exceed the limits of physical endurance of any teacher.

To cope with these management problems, a variety of organizational schemes has been utilized. One of the simplest is permanent intra-class groups. This technique customarily involves the use of an ability criterion to separate the class into, say, the A, B, and C sections. In theory, the idea is to construct a number of nearly homogeneous subgroups in a heterogeneous class. This compromise preserves the class structure within the subgroups. The teacher can assign work on the basis of the ability of the group, maintain contact with the sections much in the same manner as he would with the whole class, and continue to evaluate students' progress in a conventional manner. Some problems occur in ranking and grading students, of course. For example, should the best student in the least talented group be given a higher grade than say the least able person in the most talented group? Some teachers determine the maximum grade to be given to any one group, e.g., A, B, or C, respectively. However, this grading policy tends to restrict students to certain performance levels. One apparent remedy is to make the intra-class groups flexible, thereby allowing a student to shift from one group to another depending on the performance in a given group.

Another grouping technique is that of departmentalizing the school. In a departmentalized school, a single student could be placed in groups of various ability levels depending on the subject and the level of performance in that subject. This type of arrangement, though meeting some of the desirable democratic principles of classroom organization, begins to produce considerable stress on the teacher or the team of teachers. Keeping track of the problems and progress of individuals, preserving good articulation between groups, and setting fair and consistent criteria for intra-class group transition are not easy tasks.

A somewhat more involved technique of grouping is employed in non-graded schools and in schools with inter-class grouping. In these cases, the option exists of fixing grouping arrangements for a given period of time or allowing them to remain flexible. In a flexible arrangement, a child is placed in a particular class if certain criteria levels are met. The simplest criteria to use are those of performance on achievement tests; however, a variety of sociometric and psychometric measures may be used in addition to purely intellectual parameters or subjective judgment by the teacher. The particular combination of criteria chosen depends on many factors: the school—its organization and administration, the teacher or team, the objectives of the curriculum, the type of children, and so on.

There is no one grouping arrangement that is best for all children, all teachers, all curricula, and all schools. The design used in meeting individual differences must be highly specific to each of these considerations. It is a poor strategy to adopt totally and uncritically someone else's scheme. The chances of success in such a situation are small.

THE INSTRUCTIONAL VARIABLES

Once a grouping arrangement has been selected, decisions have to be made with respect to the kind of instructional differences one wishes to provide for groups of pupils. There are three basic methods of adjusting the mathematics curriculum; these methods, incidentally, are not mutually exclusive, but are mutually consistent.

The first and most obvious way is that of pacing. In general, one might expect brighter groups of students to learn faster. This assumption, however, does not always prove to be valid. The consideration basic to the notion of pacing is that the child's behavior be adaptive. In other words, ideal pacing for each group of students should vary over time in such a way as to provide optimal progress toward educational goals. Some children work effectively at rather high rates of speed, whereas others benefit from more contemplative rates. In either case, the ideal rate does vary with time and is not a simple function of intelligence or achievement. Thus, one would expect considerable variation among individuals within groups.

Pacing may be externally controlled by the demands of the curriculum or by the teacher. On the other hand, it may be self-determined by the pupil. The former method maintains teacher control and minimizes management problems. However, it runs the risk of not producing optimum learning effects for all concerned. Self-pacing, on the other hand, may be effective with students who have some insight into their own learning mechanism and optimum pacing. Other students who, for reasons of motivation or maturity, are not so gifted may work too fast for their own good (or not at all). Recent experiments with primary grade children have indicated that considerable acceleration is possible for gifted children. Even in such a select group, enormous differences in learning periods are encountered. Whether acceleration is *desirable* or not is a decision that each school and its staff must make. If acceleration is encouraged, a school should take into consideration articulation problems from grade level to grade level, if these exist, or from school to school. Care must be exercised in designing the long range components of an individualized program.

A second instructional variable that can be adjusted to the individual's needs is that of level of difficulty. In some mathematics textbooks, exercises have been arranged according to presumed levels of difficulty for this purpose. A problem that often occurs when this method is used

by schools is that the factors determining difficulty are poorly understood. In addition, students are often pressured into working toward "top-level" groups by their own drives or by their parents for reasons not always consistent with effective learning.

The third instructional variable, closely related to variations in pacing and in level of difficulty, is enrichment activities. A child who works effectively at a high speed and completes even difficult exercises with dispatch is often led to study other areas of mathematics not normally a part of the standard curriculum. Enrichment activities have been used more frequently in recent years, not only for the mathematically talented, but also, to some degree, for students anywhere on the spectrum of individual differences.

An important influence on enrichment activities in many elementary classes has been the "mathematics corner" or "mathematics learning center." Many elementary teachers have found that individual differences and needs are detected and best met if a rich variety of instructional materials is available to individual learners. Learning centers facilitate enrichment activities when they include an abundance of such objects as counters, counting frames, measuring instruments, place value charts, geometric objects, and number lines.

In a broad sense, the adjustment of these instructional variables presents the possibility of an indvidualized curriculum for intra-class groups, inter-class groups, and, in the extreme, for single individuals. Little is known at this time about how such a program could be administered or, in fact, whether children could be expected to select topics for study as they are expected to pace themselves or to select fitting levels of difficulty at which to work. However, trends in mathematics education are breaking down the uniformity and conformity which characterized traditional programs. Hand-in-hand with this development has come the necessity of rethinking the nature and objectives of the mathematics curriculum. Experiments with individualized curricula will hopefully yield a theory of individualization which will subsume the special cases of enrichment and acceleration.

SUMMARY

The adjustment of mathematics programs to individual abilities is a challenging task. Success requires the effective cooperation of all concerned with the instructional task. The school must meet its responsibilities with imaginative administration, especially in developing a well-integrated and articulated curriculum, in providing assistance in classroom management, in utilizing teachers effectively, and in devising grouping schemes. Teachers of mathematics must meet their responsibilities by improving their subject matter competence so as to be more capable of evaluating pupil progress, of assigning students to groups, of advising students on enrichment materials, and of monitoring effective working speeds of pupils.

School systems should do their best to provide the material assistance necessary for the program. Individualization may be fostered through the use of adequately supplied mathematics learning centers in each classroom. Audio-visual aids are frequently useful in programs designed to meet individual differences as is the use of instructional TV or closed circuit TV. In some cases programmed instructional materials or computer assisted instruction may be found beneficial. Good library resources are necessary, both for enrichment activities and for accelerated students.

In large-scale programs, it may be necessary to turn to modern technology for management assistance. Efforts have been made in the last few years to provide computer assistance for the classroom management aspects of programs that are adjusted to abilities of individual students.

Evaluation and Testing

Evaluation is a continuous and integral part of the successful school mathematics program. It has many facets and necessarily involves all staff members who are associated with the teaching of mathematics. Certain phases of evaluation may be objective in nature, but the process must accommodate the many value judgments that are both necessary and desirable. The techniques of evaluation must be both formal and informal and must include procedures for appraising the development of interests and attitudes, as well as skills and understandings.

EVALUATING THE MATHEMATICS CURRICULUM

The purposes of evaluating the mathematics curriculum are two-fold, but are very intimately related. The first is to measure the extent to which the experience and behavior of the children reflect the content and instructional objectives of the mathematics curriculum. The second is to use this information in making adjustments in the instructional program that will provide for even greater realization of the objectives. As curricula are revised and the objectives are perhaps modified, the evaluation procedures must be adjusted so as to measure the intended outcomes of the revised curricula.

Published or standardized objective tests continue to be an important source of information about the effectiveness of the mathematics program. Tests must be carefully chosen. If the test scores are to have validity, they should measure only those abilities that the curriculum is designed to develop. As the objectives of the mathematics curriculum are broadened, the task of selecting appropriate standardized tests becomes even more important and increasingly difficult. The once popular test with major emphasis on computational skills is of little value when used alone to evaluate today's mathematics curriculum with its varied and extended objectives. Test scores take on significance only as they are considered along with other measurements of the effectiveness of the curriculum.

The majority of standardized tests are of the objective, short answer type, but the nature of these tests does not necessarily limit them to measuring only discrete, factual

information. Properly designed objective tests can measure mathematical insight and originality, as well as levels of learning of difficult concepts.

Regardless of how appropriate a standardized test may be in evaluating the mathematics curriculum, its use is justified only if the results can be employed in improving instruction. System-wide analysis of scores is certainly one legitimate way of using test scores for evaluating and ultimately improving the mathematics instruction. When tests are used for this purpose, the question always arises as to what standards or norms should be used. Many school systems are less interested in comparing their children with other groups than they are in studying the performance of the children within their own system over a period of years. In this case, school systems should be encouraged to develop local norms. The use of a national or regional norm is not necessarily to be discouraged; however, it should be used only in the light of complete information with regard to its development.

It is also important that standardized test results be used to improve instruction within each teacher's classroom. Certainly these results embody significant information that will help increase the effectiveness of any teacher's instruction. The standardized test results may also be of some value in analyzing the instructional needs of the individual child.

Whatever use be made of standardized tests, schools must be careful that the material in the test does not dictate what is taught in the classroom. The pitfall of teaching to the test must be avoided at all costs! Some of the motivation for teaching to the test can be eliminated if everyone recognizes that a given test, when administered, does not measure recent achievement alone. Tests measure achievement and understandings developed by a child over several years. All teachers that a child has had must share both in the success or lack of success as reflected in a given test result.

Few questions associated with the evaluation process have definite "yes" or "no" answers. Contemplation of questions such as the following may, however, give some insight into the evaluation of the mathematics curriculum.

a) Does the curriculum differ markedly from that of most school systems?
b) Does the curriculum develop and maintain interest in mathematics?
c) Is the curriculum flexible enough to meet the needs of each child?
d) Does the curriculum encourage creativity, originality, and mathematical experimentation by the children?
e) Does the curriculum develop adequate facility with computational skills?
f) Does the curriculum give proper emphasis to the unifying concepts and structure of mathematics?
g) Does the curriculum prepare the children to use their mathematical skills and knowledge outside the mathematics class?
h) Does the curriculum present mathematics as an enjoyable and fascinatingly alive subject?

The process of evaluating the curriculum must not focus on simply whether the children are learning what is being taught. Attention must be given to evaluating whether that which is being taught is that which should be taught. It must also be recognized that due to the varied backgrounds and abilities of the children, flexibility in the expected minimum levels of achievement is necessary.

EVALUATION OF THE CHILD'S MATHEMATICAL DEVELOPMENT

Although evaluation of the effectiveness of the entire elementary school mathematics curriculum is of prime importance, the evaluation of the learning experiences and achievement of each child as an individual is of at least equal significance. In reality, such evaluation is the final test of the effectiveness of a mathematics curriculum in a society which places ultimate value on the worth of every individual.

The evaluation of a child's growth in developing mathematical concepts and abilities, like the evaluation of the entire mathematics curriculum, necessarily involves a variety of techniques and procedures. This appraisal must include the many aspects of learning beyond the acquisition of facts and skills. Not only must educators know whether the pupil is succeeding or failing, but they must also strive to ascertain the level of success with respect to each of the objectives of the mathematics program.

The evaluation of a child's progress must be deliberate and systematic, yet it must allow for the measurement of growth (or lack of growth) at any time when it can be observed. It matters little whether the evaluative procedure is the interview, a conversation or exchange of ideas between teacher and child, a standardized or teacher-made test, or the observation of the behavior of the child in a mathematical situation. The relevance of an evaluation is determined by the degree to which it measures the pupil's growth toward realization of specific objectives of the mathematics curriculum.

With reasonable care and a small amount of ingenuity, the teacher can devise test questions or activities to evaluate progress toward the realization of any of the content or behavioral objectives of the mathematics program. The following are offered as illustrations of possible evaluative activities.

1. In the lower elementary grades (even at the pre-number level), the child's appreciation for number properties can be appraised by providing two collections of objects and asking whether there are as many objects in one collection as in the other. If the collections of objects are so large that they cannot be mentally matched, understanding of the role of one-to-one correspondence between sets with the same number property can be appraised.
2. The appreciation of the kindergarten child for the number property of a set can be observed by asking which of two sets of objects has the greater number of objects, when one set has a large number of small objects and the other has a small number of much larger objects.
3. The ability of the second or third grade child to generalize relationships can be evaluated by stating that 5 + 7 is 12, 15 + 7 is 22, and 25 + 7 is 32, and by then asking what 85 + 7 equals. The desired generalization has been made if "92" is an immediate response. The generalization probably has not been made if the response is slower or if there is evidence of counting or adding.
4. This example is one way of presenting the second or third grade child with an opportunity to show an understanding of the distributive property. State, perhaps orally, that Jane has seven marbles and Ted has three marbles. Then ask how many marbles they would have altogether if they each had two times as many marbles. The child who understands that the

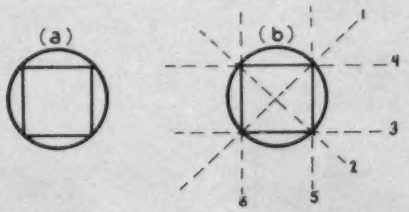
correct answer may be found by computing (2 x 7) + (2 x 3) or by computing 2 x (7 + 3) has demonstrated understanding of the distributive law. The property is understood even if the student is not able to verbalize the relationship expressed by it.

5. In the middle grades, evidence of the child's ability to generalize may be obtained in the following way. Point out that: $(12)^2 = 144$ and $(13)^2 = 144 + 24 + 1 = 169$; and that $(20)^2 = 400$ and $(21)^2 = 400 + 40 + 1 = 441$. Observe that $(40)^2 = 1600$ and ask what $(41)^2$ equals. A further generalization would be required to determine what $(39)^2$ equals.
6. In the middle grades, understanding of the associative property of multiplication can be appraised by observing how the student computes (13 x 5) x 2. The student who understands the associative property will probably recognize that (13 x 5) x 2 = 13 x (5 x 2) and will then be able to carry out the operation mentally by using 13 x 10. Such problems when given as mental exercises can be used both to develop and to test an understanding of this basic property of multiplication.
7. The ability to translate an English statement into a mathematical sentence is a vital part of solving some problems. This aspect of problem solving can be evaluated in the middle grades by giving the child the following exercise.

   Choose the mathematical phrase or sentence that best expresses the English phrase or sentence and write the letter of your choice in the blank preceding the English phrase or sentence.

__the number is five	A. $5 \times \square = 5$
__five less than the number	B. $\square + 5$
__five more than the number	C. $5 \times \square$
__five times the number	D. $\square - 5$
	E. $5 - \square$
	F. $5 + \square$
	G. $5 \times \square + \square$
	H. $\square = 5$

8. The following is an example of a type of problem situation that can be presented to students in grades 4 to 8. Provide the child with a piece of waxed paper on which a circle has been marked. By folding the paper, a square that just fits inside the circle, as in (a) below, is to be formed.



   The paper can be folded as many times as desired and in any way. Sketch (b) indicates at least one way in which successive folds can be made so as to carry out the assignment. Observing the child in the process of solving such a problem, even if a simple one, provides the teacher with real insight into a child's ability to bring a variety of knowledge to bear on a new or strange situation.
9. The following type of performance exercise could be used to assess the child's appreciation for the relationship between similar figures. Give the student a triangular piece of paper and ask that it be cut so as to form a triangular piece of paper similar in shape to the original triangle, but with one-half the perimeter. The child who proceeds to cut the original triangular shape from the midpoint of one side to the midpoint of another side will be displaying an understanding of the relationship between similar triangles.
10. A multiple choice exercise such as the following might be useful in appraising a student's understanding of why division by zero is not possible.

    Which of the following is a correct statement about the problem $6 \div 0$?

    A. The answer to $6 \div 0$ is 6.
    B. The answer to $6 \div 0$ is 0.
    C. There is no answer to $6 \div 0$ because there is no number that when multiplied by 0 gives 6 as a result.
    D. Either 0 or 6 is correct so there is no one answer to the problem $6 \div 0$. It depends on the answer you want to use.

The above examples of evaluation procedures are not intended to be taken as models, but only as illustrations of the wide variety of techniques that can be used. It is frequently possible to present, with slight modification, the same basic question or test situation to the pupil in the form of an oral, a written, or a performance experience. It may be necessary to adjust the evaluation procedure to the pupil, for not all pupils can be expected to achieve the same level of understanding of mathematical ideas. In order to determine the level of understanding a child has reached with respect to a given concept, one may have to devise a series of questions, each intended to measure a different level of understanding. One may also employ a more lengthy question that calls for several immediate responses, each requiring a different level of understanding. In any event, the child's background and potential must be considered, both when deciding which evaluation procedures to use and when interpreting the degree of success the pupil has experienced.

An increasing number of rating scales and check lists are available to help the teacher in evaluating the pupil's attitudes toward mathematics as well as his interest in the subject. For the most part, these instruments are of unproven reliability and validity. Consequently, the teacher's value judgment remains a significant source of evaluation in this area. These factors along with the pupils' enjoyment of mathematics may be appraised by observing their behavior when confronted with mathematical situations. Are the children number conscious in their recreational reading or in their reading of other school subjects? Are they interested in recreational mathematics? Are they fascinated by mathematical puzzles? Do they display an inquiring attitude toward mathematics by raising questions about numbers, operations, or their properties that indicate they are thinking about concepts beyond those which they have studied?

Children in the lower grades can be asked questions about what they like best or least about numbers, adding, circles, or counting. In the upper grades, the children can be asked to write a short essay or paragraph on what they think about when someone says *mathematics* or on what they like best or least about mathematics. At times the teacher should request that pupils leave their essays or paragraphs unsigned so as to encourage a freer response and thus to obtain a more accurate picture of the attitude of the class as a whole.

Today's elementary school mathematics program continues to have as an objective, the development of computational skills. Both commercial and teacher-made objective tests are effective instruments in evaluating the degree to which this objective is achieved. Tests should be used which have a wide variety of items, each designed to test a particular skill. In this respect, there is some advantage in using commercial tests in that more care and

perhaps greater and more varied test writing experience are employed in designing a given test.

Care must be exercised in interpreting a test score when it is being used to evaluate a child's progress. A single test score has meaning only when it is viewed in relation to other measures of achievement. Patterns formed over a period of time by test scores are also useful in assessing a child's mathematical development.

Evaluation of the mathematics curriculum or the mathematics achievement of an individual child must not be separated from the teaching of mathematics. Evaluation is an integral part of teaching and must be continuously brought into play in the process of directing and re-directing learning experiences. It is a vital part of the ongoing program of curriculum improvement. Evaluation in elementary school mathematics must not be viewed as a process of appraising a finished product or a spent program. Instead, it must be recognized as the process by which a continuous record is maintained of the pulse rate of every aspect of both the mathematics curriculum and the experiences of the children under the curriculum.

Criteria for Program Development

AN EXEMPLARY MATHEMATICS PROGRAM

The purpose of the study of mathematics in our schools is to help all students to think functionally in varying degrees about quantitative and special matters. In order to achieve this end, each student needs to acquire a certain degree of skill in the manipulation of a number of mathematical facts and concepts. He is then able to develop a sound understanding of mathematical relationships, generalizations, and principles which support the fundamental skills.

As the fundamental laws of mathematics are observed, developed, and used, students at an early age begin to see that they have a medium in which they can "think" about a vast number of problems and situations. This early beginning, though incomplete and imperfect, points clearly to the following fundamental objectives of teaching mathematics in our schools: (1) the development of an understanding of the basic numerical and geometrical aspects of nature as distinct from words and symbols used to name these aspects; (2) an understanding of the fundamental properties of numbers with their respective laws of operation; (3) the ability to use the ideas of mathematics to explore new problems and their solutions; (4) an appreciation of the role of mathematics in the scientific and social aspects of our culture; and (5) a realization of the importance of mathematics as a segment of human knowledge which is worthy of study in its own right and which provides man with a means of seeking "truth."

From these objectives, it is possible to list the expected outcomes of an exemplary mathematics program:

- The most important goal of mathematics instruction is the development of students' ability to solve problems. The term "problem" is used to mean a question that is posed for which an answer is not immediately available and for which the student desires to find a solution. It is assumed that the student is required to use more than immediate recall or previously established patterns of action to find a method for arriving at a conclusion. In particular, the skills of mathematics and the correct applications of mathematical principles and methods are essential to this process. Emphasis should be placed on those problems which arise from real life settings.
- Students develop the facility to apply previously learned mathematical principles, skills and methods to real life situations so that some problems do not arise.
- Students attain the prerequisites for continued formal study (and/or independent study) in mathematics, related fields and other subjects using mathematics.
- Students realize that while mathematics can be a useful tool for solving problems and making decisions, it is also recognized as a science, an art form and a language. As a science, the student is involved with a study of patterns and also looks at mathematics as a unified structure. As an art form, the student is given the opportunity to be both creative and original. The language of mathematics enables the student to comprehend and communicate ideas which would otherwise be beyond his communication capability.
- Young children maintain the natural interest they have in mathematics with the resulting desire to continue study. The student recognizes mathematics as an alive, enjoyable, game-like discipline with profound implications for the future of society.

In order to accomplish these outcomes, there are identified processes and methods that are typical of exemplary mathematics programs.

- Those characteristics related to the means of achieving the output goals are centered around the notion of success-oriented instruction. Individual pupil success in the learning of mathematics can occur only when differences between pupils are taken into account. The instruction is individualized along three dimensions: time, content and mode. The latter dimension must feature active learner involvement, initiating with concrete manipulative experiences and progressing, as readiness permits, to activities dealing with concepts of a more abstract nature. The self-study mode must not dominate at the elementary level. Mathematics is "talked" as well as "worked," through liberal teacher-pupil and pupil-pupil interaction via group activities and discussion. Cyclic reinforcement and/or content spiraling is provided as necessary.
- The ideal mathematics program possesses a built-in accountability system capable of retrieving diagnostic information on individual pupil progress and continuously assessing overall program adequacy and efficiency.

- In the final analysis, however, the student remains the most important instructional variable and as such, plays a most important role in the learning environment. The teacher's competence in subject matter and in classroom management skills are indispensable if they are to interact effectively with pupils.

CURRICULUM DEVELOPMENT AND IMPLEMENTATION

Most curriculum development activity in Wisconsin results not in a curriculum which is implemented in the classrooms of the district, but in the selection of a textbook series. That textbook series becomes the de facto curriculum for the district. The role of teachers in curriculum development then is twofold: 1) to identify the criteria by which the textbook is selected, and 2) to make recommendations for the uniform implementation of the textbook in the classrooms of the district. The sample criteria checklist which follows is designed to assist teachers in the textbook selection process. The checklist is based on the philosophy and instructional objectives expressed in these Guidelines to Mathematics: K-8 as well as the recommendations contained in the National Council of Supervisors of Mathematics "Position Paper on Basic Mathematical Skills," and the National Council of Teachers of Mathematics' An Agenda for Action: Recommendations for School Mathematics of the 1980s.

Once a textbook is identified, a consistent and uniform mathematics curriculum results only if the textbook is used according to a district-wide plan. That plan should be designed to provide each pupil with a continuous program throughout grades K-8, with a balanced emphasis and no conspicuous omissions of important topics or concepts. To insure this, all teachers should teach from all chapters of the book at all grade levels. Time limitations should dictate that topics are deleted from within chapters, rather than deleting certain chapters in their entirety. The curriculum committee should designate topics within each chapter as essential or optional, and design an articulation strategy which enables all teachers to understand what the curriculum is, as well as their individual role in its implementation.

These guidelines have suggested grade levels for the *introduction* of the major mathematical conepts. They do not provide a map for the subsequent development leading to mastery, nor for the periodic review and retention activities necessary once mastery is achieved. An exemplary program will include systematic review and retention activities throughout the K-8 program.

SAMPLE CRITERIA CHECKLIST

Name of Program	Publisher	Grade	Teacher

The evaluation column, which comes either to the left or right of the criteria statements, is:

Rating				
Low				High
1	2	3	4	5

Sometimes a four-point scale is preferred because it eliminates the possibility of a completely neutral response.

1. Philosophy of Program
   The program
   a) is organized around problem solving and integrates problem solving into all activities at all grade levels.
   b) stresses the applications of mathematics in everyday situations.
   c) stresses estimation and approximation in mental calculation and checking the reasonableness of all results.
   d) introduces concepts using appropriate concrete examples and makes a transition through pictorial representations to abstract representations and processes as appropriate.
   e) uses a spiral development.
   f) uses calculators and computers to develop mathematical concepts as appropriate.
   g) considers the mathematical needs of the students as adults in the twenty-first century.
2. Key Mathematical Content and Instructional Objectives of Program
   A. Problem Solving
      The program
      a) provides continuous emphasis on problem solving at all grade levels.
      b) develops recognition of appropriate algorithms for applying mathematics to everyday situations.
      c) develops a variety of techniques for solving problems having no standard algorithm.
      d) includes problems with more than one correct answer.
   B. Measurement
      The program
      a) develops measurement concepts through the measuring of objects and quantities.
      b) provides opportunities to measure time and money.
      c) develops and applies both non-standard and standard units for length, area, volume, capacity, and angle measure, including metric units where appropriate.
      d) provides experiences with approximate measures.
      e) develops a knowledge of equivalent measures.
      f) creates an awareness of the approximate nature of measurement and the error inherent in the measurement process.
      g) develops area formulas for squares, rectangles, parallelograms, and triangles.
   C. Computation
      The program
      a) develops an understanding of numbers and numeration, including place value.
      b) stresses the meaning of the operations of addition, subtraction, multiplication, and division as applied to whole numbers and fractions.
      c) develops computational algorithms through concrete and pictorial procedures.
      d) uses computational skills appropriately in problem solving, including applying mathematics to everyday situations.
      e) emphasizes estimation and alertness to the reasonableness of results.
      f) develops the ability to use mental computation, estimation, pencil and paper computation, and computation with aids such as tables, calculators, and computers, and develops the ability to determine which method of computation is appropriate for a given situation.
      g) emphasizes the inverse relationship between addition and subtraction; between multiplication and division.
      h) provides opportunities for periodic review of previously learned skills.
   D. Geometry
      The program
      a) develops recognition of common plane and solid figures.
      b) contains activities leading to the identification of properties of common plane and solid figures.
      c) provides manipulative and conceptual experiences with points, lines and planes at appropriate grade levels.

d) develops recognition of parallel and perpendicular lines and planes.
e) includes activities which develop symmetry concepts and recognition of symmetry in plane and solid figures.
f) includes activities which develop congruence concepts, including congruence of segments, angles, polygons, and solid figures.
g) includes activities which develop similarity concepts and recognition of similarity in plane and solid figures.

E. Statistics
The program
a) introduces graph building and graph reading at the beginning (grade K or 1) and continues throughout (using pictographs, bar graphs, circle graphs, or broken line graphs as appropriate).
b) includes activities which require reading, interpreting, and constructing tables and charts for the communication of information.
c) provides activities with numbers on a number line, and extends to two dimensional and three dimensional coordinate systems.
d) develops concepts of probabilities of simple events.
e) provides activities in which the probability of an event is estimated from statistical data.
f) introduces common measures of dispersion and central tendency (range, mean, median, mode) and their value in decision making.

F. Calculators and Computer Literacy
The program
a) includes calculator activities at all grade levels.
b) includes instructional activities which develop computer literacy[1] and which do not require student interaction with a computer.
c) includes instructional activities which develop computer literacy through direct interaction with computers.
d) provides appropriate opportunities to use calculators and computers in problem solving.

3. Methodology Used In and Suggested For Program
The program provides
a) mathematical concepts correctly and consistently used in explanations.
b) a variety of approaches in developing a topic.
c) a basic skills development and maintenance program.
d) adequate material for written assignments and oral exercises.
e) an emphasis on estimation and "trial-and-error" techniques in problem solving.
f) mental arithmetic activities.
g) independent thinking activities.
h) exercises graduated in difficulty.
i) consideration of individual differences.
j) clarity of examples.
k) supplementary activities for high achievers.
l) supplementary remedial activites for low achievers.

4. Supplementary Aids to Program
a) The teacher's manual contains mathematical background for content of program, rationale for inclusion of topics, suggestions for teaching, and suggestions for enrichment activities.
b) A pupil evaluation program based on criterion referenced testing keyed to program objectives is available.
c) Computer software is available for pupil drill and practice, concept development, pupil testing and record keeping.
d) Manipulative materials for use with program activities are available.
e) The publishing company provides consultants, films, or other inservice aids.

A textbook adoption committee should review and revise the criteria checklist before using it as a basis for textbook selection. Another activity which should take place prior to the application of the criteria to the textbooks being considered is a discussion of the interpretation of the scoring for each textbook series. It would be a mistake to automatically select the text with the highest total score since no attempt has been made to balance the list. It is not the case that each suggested criterion is equally as important as all others. Each evaluator should probably attempt to give each textbook a composite score in each of the categories: philosophy, problem solving, measurement, computation, geometry, statistics, calculators and computer literacy, methodology, and supplementary aids. A low score in any one of these categories is probably sufficient to eliminate a program from consideration.

[1]The most detailed definition of computer literacy to date appears in "Computer Literacy - What Is It?" The Mathematics Teacher, Volume 73, Number 2, February 1980, pp. 91-96.

An Agenda For Action: Recommendations For School Mathematics of the 1980s

PREFACE

In the 1960s there was considerable ferment in mathematics curriculum and instruction. Although public attention was focused on the more visible attempts at program revision, we are aware two decades later that change was more apparent than real.

In the 1970s the concern of the public was directed toward problems evidenced almost exclusively in test scores. Schools have responded to this concern in a variety of ways, but a clear-cut and carefully reasoned sense of direction that looks toward the future has been lacking.

We recognize as valid and legitimate the role of public opinion in the determination of educational goals. But this philosophy is predicated on a well-informed public. Thus, the National Council of Teachers of Mathematics, as an organization of professional educators, has a special obligation to present its responsible and knowledgeable viewpoint of the directions mathematics programs should be taking in the 1980s.

These recommendations represent both realism and responsibility. They are realistic in their attention to hard data. We are fortunate to have more information about mathematics classroom practice than we have ever had. This useful information comes principally from a series of studies funded by the National Science Foundation and from two mathematics assessments of the National Assessment of Educational Progress.

The recommendations are responsible to the profession and to the public since they represent a very broad base of belief about objectives and priorities. The Council, funded by the National Science Foundation, conducted an extensive survey of the opinions of many sectors of society, both lay and professional. The project was called Priorities in School Mathematics (PRISM).

Such opinion surveys do not in themselves generate recommendations, but a professional organization, if it is to be responsible, must give them serious consideration as it develops its best-considered advice to society concerning future directions for educational programs.

These recommendations are not the end of our efforts but a beginning. They represent an agenda for a decade of action, and we call on all interested persons and groups to join us in a massive cooperative effort toward better mathematics education for all our youth.

April 1980

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The National Council of Teachers of Mathematics recommends that:

1. problem solving be the focus of school mathematics in the 1980s;
2. basic skills in mathematics be defined to encompass more than computational facility;
3. mathematics programs take full advantage of the power of calculators and computers at all grade levels;
4. stringent standards of both effectiveness and efficiency be applied to the teaching of mathematics;
5. the success of mathematics programs and student learning be evaluated by a wider range of measures than conventional testing;
6. more mathematics study be required for all students and a flexible curriculum with a greater range of options be designed to accommodate the diverse needs of the student population;
7. mathematics teachers demand of themselves and their colleagues a high level of professionalism;
8. public support for mathematics instruction be raised to a level commensurate with the importance of mathematical understanding to individuals and society.

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Recommendation 1

PROBLEM SOLVING MUST BE THE FOCUS OF SCHOOL MATHEMATICS IN THE 1980s

The development of problem-solving ability should direct the efforts of mathematics educators through the next decade. Performance in problem solving will measure the effectiveness of our personal and national possession of mathematical competence.

Problem solving encompasses a multitude of routine and commonplace as well as nonroutine functions considered to be essential to the day-to-day living of every citizen. But it must also prepare individuals to deal with the special problems they will face in their individual careers.

Problem solving involves applying mathematics to the real world, serving the theory and practice of current and emerging sciences, and resolving issues that extend the frontiers of the mathematical sciences themselves.

This recommendation should not be interpreted to mean that the mathematics to be taught is solely a function of the particular mathematics needed at a given time to solve a given problem. Structural unity and the interrelationships of the whole should not be sacrificed.

True problem-solving power requires a wide repertoire of knowledge, not only of particular skills and concepts but also of the relationships among them and the fundamental principles that unify them. Each problem cannot be treated as an isolated example. This recommendation looks toward the need to solve problems in an uncertain future as well as here and now. As such, mathematics needs to be taught as mathematics, not as an adjunct to its fields of application. This demands a continuing attention to its internal cohesiveness and organizing principles as well as to its uses.

Recommended Actions

1.1 *The mathematics curriculum should be organized around problem solving.*

- The current organization of the curriculum emphasizes component computational skills apart from their application. These skills are necessary tools but should not determine the scope and sequence of the curriculum. The need of the student to deal with the personal, professional, and daily experiences of life requires a curriculum that emphasizes the selection and use of these skills in unexpected, unplanned settings.
- Mathematics programs of the 1980s must be designed to equip students with the mathematical methods that support the full range of problem solving, including—
  - —the traditional concepts and techniques of computation and applications of mathematics to solve real-world problems, the rational and real number systems, the notion of function, the

use of mathematical symbolism to describe real-world relationships, the use of deductive and inductive reasoning to draw conclusions about such relationships, and the geometrical notions so useful in representing them;

—methods of gathering, organizing, and interpreting information, drawing and testing inferences from data, and communicating results;

—the use of the problem-solving capacities of computers to extend traditional problem-solving approaches and to implement new strategies of interaction and simulation;

—the use of imagery, visualization, and spatial concepts.

- Mathematics programs should give students experience in the application of mathematics, in selecting and matching strategies to the situation at hand. Students must learn to—
  - —formulate key questions;
  - —analyze and conceptualize problems;
  - —define the problem and the goal;
  - —discover patterns and similarities;
  - —seek out appropriate data;
  - —experiment;
  - —transfer skills and strategies to new situations;
  - —draw on background knowledge to apply mathematics.
- Fundamental to the development of problem-solving ability is an open mind, an attitude of curiosity and exploration, the willingness to probe, to try, to make intelligent guesses.
- The curriculum should maintain a balance between attention to the applications of mathematics and to fundamental concepts.

1.2 ***The definition and language of problem solving in mathematics should be developed and expanded to include a broad range of strategies, processes, and modes of presentation that encompass the full potential of mathematical applications.***

- Computational activities in isolation from a context of application should not be labeled "problem solving."
- The definition of problem solving should not be limited to the conventional "word problem" mode.
- As new technology makes it possible, problems should be presented in more natural settings or in simulations of realistic conditions.
- Educators should give priority to the identification and analysis of specific problem-solving strategies.
- Educators should develop and disseminate examples of "good problems" and strategies and suggest the scope of problem-solving activities for each school level.

1.3 *Mathematics teachers should create classroom environments in which problem solving can flourish.*

- Students should be encouraged to question, experiment, estimate, explore, and suggest explanations. Problem solving, which is essentially a creative activity, cannot be built exclusively on routines, recipes, and formulas.
- The mathematics teacher should assist the student to read and understand problems presented in written form, to hear and understand problems presented orally, and to communicate about problems in a variety of modes and media.
- The mathematics curriculum should provide opportunities for the student to confront problem situations in a greater variety of forms than the traditional verbal formats alone; for example, presentation through activities, graphic models, observation of phenomena, schematic diagrams, simulation of realistic situations, and interaction with computer programs.

1.4 *Appropriate curricular materials to teach problem solving should be developed for all grade levels.*

- Most current materials strongly emphasize an algorithmic approach to the learning of mathematics, and as such they are inadequate to support or implement fully a problem-solving approach. Present textbook problems tend to be easily categorized and stylized and often bear little resemblance to highly diversified, real-life problems. They do not permit the full range of strategies and abilities actually demanded in realistic problem contexts.
- The potential of computing technology for increasing problem-solving ability should be explored and exploited by the development of creative and imaginative software.

1.5 *Mathematics programs of the 1980s should involve students in problem solving by presenting applications at all grade levels.*

- Applications should be presented that use the student's growing and changing repertoire of basic skills to solve a multitude of routine and commonplace problems essential to the day-to-day living of every citizen.
- Applications of mathematics to other disciplines such as the social sciences, business, engineering, and the natural sciences should be presented.
- The enormous versatility of mathematics should be illustrated by presenting as diversified a collection of applications as possible at the given grade level.
- At the college level, courses in mathematics and the mathematical

sciences should give prospective teachers experiences that develop their capacities in modeling and problem solving.

1.6 *Researchers and funding agencies should give priority in the 1980s to investigations into the nature of problem solving and to effective ways to develop problem solvers.*

- Support should be provided for—
  - —the analysis of effective strategies;
  - —the identification of effective techniques for teaching;
  - —new programs aimed at preparing teachers for teaching problem-solving skills;
  - —investigations of attitudes related to problem-solving skills;
  - —the development of good prototype material for teaching the skills of problem solving, using all media.

Recommendation 2
THE CONCEPT OF BASIC SKILLS IN MATHEMATICS MUST ENCOMPASS MORE THAN COMPUTATIONAL FACILITY

There must be an acceptance of the full spectrum of basic skills and recognition that there is a wide variety of such skills beyond the mere computational if we are to design a basic skills component of the curriculum that enhances rather than undermines education.

We recognize as valid and genuine the concern expressed by many segments of society that basic skills be a part of the education of every child. However, the full scope of what is basic must include those things that are essential to meaningful and productive citizenship, both immediate and future.

The agreement among parents, educators, and mathematicians on the need for teaching basic skills with greater effectiveness unfortunately does not yet extend to a common understanding and acceptance of exactly what these basic skills should comprise.

Some groups narrowly limit them to routine computation at the expense of understanding, applications, and problem solving. This would leave little hope of developing the functionally *competent* student that all desire.

It must also be recognized that individual capacities, interests, and future directions might call for different emphases and different selections in matching basic skills to individual needs.

The time and energy that teachers and programs should be devoting to building beyond minimal foundations are sometimes skirted, being considered risky deviations from the minimal targets on which educators believe they will be judged. There is great pressure today to use all such time, energy, and re-

sources on overkill in the minimal target areas even though little added productivity may be achieved.

Rather than fostering a return to some acceptable common threshold of performance, the back-to-basics movement tends to place a low ceiling on mathematical competence—and this at the onset of an era in which daily life will be more deeply permeated by multiple and diverse uses of mathematics than ever before. Under these circumstances, even if improvement in rote computation takes place, a citizen who cannot analyze real-life situations to the point of recognizing what computations must be made to solve real-life problems has not entered the mainstream of functional citizenship.

It is dangerous to assume that skills from one era will suffice for another. Skills are tools. Their importance rests in the needs of the times. Skills once considered essential become obsolete, and this is likely to increase in pace and scope as advances in technology revolutionize our individual, social, and economic lives. Necessary new skills arise from the dimensions of the mathematics pertinent to an age of population explosion, space exploration, economic and fiscal complexity, and microelectronic wonders. Time and space for including these new skills in the curriculum must be purchased by eliminating the obsolete.

Insisting that students become highly facile in paper-and-pencil computations such as 3841×937 or $72\,509 \div 29.3$ is time-consuming and costly. For most students, much of a full year of instruction in mathematics is spent on the division of whole numbers—a massive investment with increasingly limited productive return. A small fraction of that time is spent on the skills of problem analysis and interpretation, which enable students to identify and set up the computations needed. For most complex problems, using the calculator for rapid and accurate computation makes a far greater contribution to functional competence in daily life.

Common sense should dictate a reasonable balance among mental facility with simple basic computations, paper-and-pencil algorithms for simple problems done easily and rapidly, and the use of a calculator for more complex problems or those where problem analysis is the goal and cumbersome calculating is a limiting distraction.

Reasonable standards of time-effectiveness and cost-effectiveness should be applied to the use of instructional time, where the criterion is the productive applicability of the learned technique to real-life problems.

Professional knowledge of future trends, industrial, financial, engineering, and scientific need, and the demands of daily life are all better arbiters of what is currently essential and what has become obsolete than our nostalgia as parents or teachers.

Recommended Actions

2.1 ***The full scope of what is basic should contain at least the ten basic skill areas identified by the National Council of Supervisors of Mathematics's "Position Paper on Basic Skills." These areas are problem solving; ap-***

plying mathematics in everyday situations; alertness to the reasonableness of results; estimation and approximation; appropriate computational skills; geometry; measurement; reading, interpreting, and constructing tables, charts, and graphs; using mathematics to predict; and computer literacy.

2.2 *The identification of basic skills in mathematics is a dynamic process and should be continually updated to reflect new and changing needs.*

2.3 *Changes in the priorities and emphases in the instructional program should be made in order to reflect the expanded concept of basic skills.*

- There should be increased emphasis on such activities as—
  - —locating and processing quantitative information;
  - —collecting data;
  - —organizing and presenting data;
  - —interpreting data;
  - —drawing inferences and predicting from data;
  - —estimating measures;
  - —measuring using appropriate tools;
  - —mentally estimating results of calculations;
  - —calculating with numbers rounded to one or two digits;
  - —using technological aids to calculate;
  - —using ratio and proportion to deal with rate problems in general and with percent problems in particular;
  - —using imagery, maps, sketches, and diagrams as aids to visualizing and conceptualizing a problem;
  - —using concrete representations and puzzles that aid in improving the perception of spatial relationships.
- There should be decreased emphasis on such activities as—
  - —isolated drill with numbers apart from problem contexts;
  - —performing paper-and-pencil calculations with numbers of more than two digits;
  - —mastering highly specialized vocabulary not useful later either in mathematics or in daily living;
  - —converting measures given in one system to corresponding measures in another system;
  - —working with tables whose usefulness as aids to calculation has been supplanted by calculators and other technological aids (e.g., numerical computations with logarithms and cologs).

2.4 *Teachers should incorporate estimation activities into all areas of the program on a regular and sustaining basis, in particular encouraging the*

use of estimating skills to pose and select alternatives and to assess what a reasonable answer may be.

2.5 *Teachers should provide ample opportunities for students to learn communication skills in mathematics. They should systematically guide students to read mathematics and to talk about it with clarity.*

2.6 *The higher-order mental processes of logical reasoning, information processing, and decision making should be considered basic to the application of mathematics. Mathematics curricula and teachers should set as objectives the development of logical processes, concepts, and language, including—*

- —the identification of likenesses and differences leading to classification;
- —understanding, making, and applying definitions;
- —the development of a feeling for informal proof including counterexamples and generalizations;
- —precise use of such language as *at least, at most, either-or, both-and,* and *if-then.*

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Recommendation 3

MATHEMATICS PROGRAMS MUST TAKE FULL ADVANTAGE OF THE POWER OF CALCULATORS AND COMPUTERS AT ALL GRADE LEVELS

Beyond an acquaintance with the role of computers and calculators in society, most students must obtain a working knowledge of how to use them, including the ways in which one communicates with each and commands their services in problem solving.

The availability of computing aids, including computers and calculators, requires a reexamination of the computational skills needed by every citizen. Some of these computational skills will no longer retain their same importance, whereas others will become more important.

It is recognized that a significant portion of instruction in the early grades must be devoted to the direct acquisition of number concepts and skills without the use of calculators. However, when the burden of lengthy computations outweighs the educational contribution of the process, the calculator should become readily available.

With the increasing availability of microcomputers at decreasing costs, it is imperative that schools play an active part in preparing students of the 1980s to live in a world in which more and more functions are being performed by computers.

Recommended Actions

3.1 ***All students should have access to calculators and increasingly to computers throughout their school mathematics program.***

- Schools should provide calculators and computers for use in elementary and secondary school classrooms.
- Schools should provide budgets sufficient for calculator and computer maintenance and replacement costs.

3.2 ***The use of electronic tools such as calculators and computers should be integrated into the core mathematics curriculum.***

- Calculators should be available for appropriate use in all mathematics classrooms, and instructional objectives should include the ability to determine sensible and appropriate uses.
- Calculators and computers should be used in imaginative ways for exploring, discovering, and developing mathematical concepts and not merely for checking computational values or for drill and practice.
- Teachers should ensure in their classroom management that the use of computers by individual students in isolated activity does not replace the critical classroom interaction of students with peers and teacher. The healthy give-and-take of group work and discussion, which promotes values of communication, cooperation, empathy, mutual respect, and much of cognitive development, remains essential.

3.3 ***Curriculum materials that integrate and require the use of the calculator and computer in diverse and imaginative ways should be developed and made available.***

- Schools should insist that materials truly take full advantage of the immense and vastly diverse potential of the new media. In particular, developers of software should be cautioned that just to use conventional material and techniques newly translated to the medium of the computer will not suffice.
- Educators should take care to choose software that fits the goals or objectives of the program and not twist the goals and developmental sequence to fit the technology and available software.

3.4 ***A computer literacy course, familiarizing the student with the role and impact of the computer, should be a part of the general education of every student.***

- In cooperation with schools and professional teacher organizations, funding agencies should support the development of courses in computer literacy for both junior and senior high school levels.

3.5 ***All mathematics teachers should acquire computer literacy either through preservice programs or through in-service programs funded by school districts in order to deal with the impact of computers on their own lives and to keep pace with the inevitable sophistication their students will achieve.***

- Colleges should provide courses for both perservice and in-service education in computer literacy, programming, and instructional uses of calculators and computers.
- Professional organizations should provide information through their various media, conferences, workshops, and seminars to aid in the in-service education of teachers in uses of the calculator and computer.

3.6 ***Secondary school computer courses should be designed to provide the necessary background for advanced work in computer science.***

- Curriculum design should provide the required foundation for those students who will be involved in careers that increasingly demand advanced computing skills and applications of computing and for those students who will go on to deeper study in frontier fields of computer development.

3.7 ***School administrators and teachers should initiate interaction with the home to achieve maximum benefit to the student from the coordinated home and school use of computers and calculators.***

- Criteria should be developed to assist parents and school personnel in their selection of home/school computing hardware.
- Professional organizations of teachers, mathematicians, and computer scientists should develop guidelines to aid schools, teachers, and parents in the selection of educational software.
- The uses of technological devices such as calculators, computers, video disks, and electronic games in the home and other out-of-school places should be anticipated. Programs should be planned that will encourage the positive and educationally beneficial use of these devices.
- As home computers come into wider use, homework should be assigned that can take advantage of their potential in problem solving.

3.8 ***Educational users of electronic technology should demand a dual responsibility from manufacturers: the development of good software to promote the problem-solving abilities of the student and, eventually, the standardization and compatibility of hardware.***

3.9 ***Provisions should be made by educational institutions and agencies to help in the necessary task of educating society's adults in computer literacy and programming.***

3.10 *Teachers of other school subjects in which mathematics is applied should make appropriate use of calculators and computers in their instructional programs.*

3.11 *Teacher education programs for all levels of mathematics should include computer literacy, experience with computer programming, and the study of ways to make the most effective use of computers and calculators in instruction.*

3.12 *Certification standards should include preparation in computer literacy and the instructional uses of calculators and computers.*

Recommendation 4
STRINGENT STANDARDS OF BOTH EFFECTIVENESS AND EFFICIENCY MUST BE APPLIED TO THE TEACHING OF MATHEMATICS

What is learned relative to a topic, how long it is retained, how readily it is applied—all these depend on the learning process the students pass through and how effectively they are engaged in that process. It is fruitless to consider topics taught apart from the way learners meet these topics.

Instructional time is a precious commodity. It must be spent wisely. Learning is a product of both the time engaged in a learning task and the quality of that engagement. Teachers must employ the most effective and efficient techniques at their command. They must apportion instructional time according to the importance of the topic, recognizing that the value of a skill or knowledge is subject to change over time.

Modern technology and educational theory and research have made accessible to today's teacher approaches, materials, and strategies that were not previously available. Teachers at all levels must learn to use this enriched variety of instructional techniques, materials, and resources to teach mathematics more effectively.

Recommended Actions

4.1 *The major emphasis on problem solving in the curriculum must be accommodated by a reprogramming of the use of time in the classroom.*

- Priority in classroom time should be devoted to involving students in meaningful problem-solving activities. Explanation, practice, and directive teaching are important but should not diminish the time necessary to achieve this priority. Requiring complete mastery of skills before allowing participation in challenging problem solving is counterproductive.

- The time spent on mathematics in elementary school programs should be increased. Higher-order skills in problem solving require more time to learn than the lower-order, narrowly mechanistic skills.
- The extent of teacher and student time devoted to certain traditional skill areas should be reduced to make room for newly emerging objectives.
- There are certain algorithmic skills (e.g., long division with multiple-digit divisors) that require a great expenditure of classroom time. A strict standard of time-effectiveness and cost-effectiveness should be applied to determine whether actual use of that technique in life outside school justifies this much expenditure of effort and time. The use of calculators has radically reduced the demand for some paper-and-pencil techniques.
- Teachers should learn effective techniques of classroom management to assess and achieve the optimal time on a task.

4.2 *School administrators and parents must support the teacher's efforts to engage students more effectively in learning tasks.*

- Local administrators should ensure uninterrupted time for the teacher to carry out the instructional program.
- Parents and administrators must support the authority of the teacher to require that students be productively engaged in learning during their class time. They should exercise reasonable sanctions against students who do not respect that authority.
- Parents and administrators should expect that teachers will assign a reasonable amount of meaningful homework to extend productively the time students are engaged in the study of mathematics. Both should use their influence to increase the likelihood that students will complete the homework as assigned.
- School budgets should provide for a range of instructional resources adequate to support a wide variety of teaching strategies.

4.3 *Teachers should use diverse instructional strategies, materials, and resources, such as—*

—individual or small-group work as well as large-group work;

—well-planned use of media, such as overhead projectors, videotapes, video disks, audio/video cassettes, computers, films, slides, television;

—the provision of situations that provide discovery and inquiry as well as basic drill;

—the use of manipulatives, where suited, to illustrate or develop a concept or skill;

—the inclusion of cyclic review of past topics (contents, skills, and ideas previously taught);

—the use of materials and references outside the classroom, such as visiting museums, using the library, visiting businesses or industries, visiting computer centers, making home television assignments.

4.4 *School districts, local and state or provincial officials, manufacturers, and publishers should take a bolder and more imaginative approach to selecting and producing educational hardware and software in order to provide for a curriculum that emphasizes problem solving.*

- Unnecessarily restrictive conditions set by school districts, state and provincial agencies, outdated ground rules of publishing houses, conservative status quo editorial policies, and the like should give way to greater openness and willingness to address future needs.

Recommendation 5

THE SUCCESS OF MATHEMATICS PROGRAMS AND STUDENT LEARNING MUST BE EVALUATED BY A WIDER RANGE OF MEASURES THAN CONVENTIONAL TESTING

The first purpose of meaningful evaluation in school mathematics should be the improvement of learning programs, teaching, and materials. Educators must evaluate to have information for sound decisions, to be accountable to their public, and to know how well they are doing. Evaluation is a part of mathematics teaching, and hence mathematics educators should be centrally involved in the evaluation process.

Evaluation is not limited to testing. It includes gathering data and interpreting the data. Testing is one source of data. There are many others. Today, many people use test scores as the sole index of the quality of mathematics programs or of the success of student achievement. Test scores alone should not be considered synonymous with achievement or program quality. A serious danger to the education of our youth is the increasing tendency on the part of the public to assume that the sole objective of schooling is a high test score. This is often assumed without the critical knowledge of what is being tested or whether test items fit desired goals.

The evaluation of problem-solving performance will demand new approaches to measuring. Certainly present tests are not adequate. In particular, the measuring of the use of problem-solving processes will demand innovative techniques. Evaluation of programs with problem-solving goals must be sensitive to the nature of those goals.

It is imperative that the goals of the mathematics program dictate the nature of the evaluations needed to assess program effectiveness, student learning, teacher performance, or the quality of materials. Too often the reverse is true: the tests dictate the programs, or assumptions of the evaluation plan are inconsistent with the program's goals.

Recommended Actions

5.1 *The evaluation of mathematics learning should include the full range of the program's goals, including skills, problem solving, and problem-solving processes.*

- The evaluation of the use of problem-solving processes must be given special attention by schools, teachers, researchers, test developers, and teacher educators.
- Assessment programs, such as the National Assessment and some state assessments, should continue to sample a wide variety of mathematics learning outcomes and should consider future as well as present needs and programs.
- The development of problem-solving skills should be assessed for each student over the entire K–12 school mathematics program.
- Minimal competencies should not be construed as an adequate measure of an individual's mathematics achievement. What is minimal for all is optimal for none.

5.2 *Parents should be regularly and adequately informed and involved in the evaluation process.*

- With mutual respect (the educator for the sincere concern and valid input of the parent and the parent for the professional expertise of the educator), school administrators, teachers, and parents should cooperate in determining educational goals and the appropriate plan for evaluation.
- Nontest evaluation methods and strategies should be discussed with parents, students, and the general public.

5.3 *Teachers should become knowledgeable about, and proficient in, the use of a wide variety of evaluative techniques.*

- Preservice and in-service teacher education should provide mathematics teachers with knowledge about, and skill in, evaluation.
- Evaluation strategies that include both test and nontest techniques should be developed and disseminated to mathematics teachers both in their initial preparatory programs and in continual in-service updating.
- Teachers should improve their diagnostic skills and their ability to structure appropriate remediation.

- Student strengths and weaknesses should be assessed on a regular basis, using a variety of measures and techniques.
- The informed judgment of teachers should be considered a vital part of the evaluation of any student.
- Teachers should be prepared in the development and use of teacher-made tests.

5.4 ***The evaluation of mathematics programs should be based on the program's goals, using evaluation strategies consistent with these goals.***

- Standardized tests should be used in program evaluations only when it can be clearly demonstrated that the test matches the goals of the program.
- The results of a test designed for purposes other than program evaluation (such as the SAT) should not be interpreted as an evaluation of mathematics programs.
- Available evaluation and testing techniques should not determine the goals and objectives of the mathematics program or the emphases of classroom instructional effort.
- Test designers should give attention to the need for more options in format than the conventional multiple-choice format. An emphasis on problem solving demands more flexibility and creativity in assessment than is possible within the restrictions of most current test formats.
- Where minimal competency tests are mandated, they should be implemented with extreme caution to assure that adverse effects on the program do not result.
- Task forces involving parents, teachers, and students should be created to monitor the effects of minimal competency programs.
- Mathematics educators should be centrally involved in the development of competency or assessment programs at local, state, and provincial levels and in the monitoring of the effects of competency testing on mathematics learning.
- Longitudinal evaluation of individual problem-solving ability should be developed. The acquisition of problem-solving skills is a long-term process and should not be evaluated solely with short-term measures.
- Test scores should not be used as the sole index of success in mathematics programs.
- Accreditation of school mathematics programs should use criteria specific to the quality of the mathematics programs rather than to conditions peripheral to content and instructional goals.

5.5 ***The evaluation of materials for mathematics teaching should be an essential aspect of program planning.***

- Textbook materials should be judged and selected in terms of the program's goals rather than vice versa.
- Instructional materials with sexist and ethnic biases should not be selected.
- The selection of tests should involve a careful review by teachers as well as administrators.
- Strategies for evaluating nonprint materials must be developed and used.

5.6 *Mathematics teachers must undergo continuing evaluation as a necessary component in improving mathematics programs.*

- Teachers should maintain an awareness of the need and the strategies for self-evaluation.
- A variety of supportive evaluation strategies, such as peer observation, supervisor observation, and videotaping, should be made available to the teacher.
- Any evaluation of mathematics teaching should be sensitive to the instructional goals and should be unique to the content, the teacher, and the class.
- If an evaluation of a teacher's effectiveness includes student performance, the measures of student performance should be consistent with instructional goals. Such evaluation should also consider that there are external factors affecting student performance that are not amenable to teacher influence.
- Judgments of teacher competency are necessary but should be made with caution, with the realization that the validity of most existing measures rests on a shaky foundation.
- Teachers and teacher educators should be centrally involved in the development of instruments for the evaluation of teaching effectiveness.

5.7 *Funding agencies should support research and evaluation of the effects of a problem-solving emphasis in the mathematics curriculum.*

- The nature of problem-solving ability suggests that longitudinal studies will be most meaningful. The more typical short-term project may force a hasty and superficial treatment of programs whose objectives must be complex, interrelated, and of a long-lasting character.

Recommendation 6

MORE MATHEMATICS STUDY MUST BE REQUIRED FOR ALL STUDENTS AND A FLEXIBLE CURRICULUM WITH A GREATER RANGE OF OPTIONS SHOULD BE DESIGNED TO ACCOMMODATE THE DIVERSE NEEDS OF THE STUDENT POPULATION

Mathematics is pervasive in today's world. Mathematical competence is vital to every individual's meaningful and productive life. It is, moreover, a valuable societal resource, and the potential of our educated citizenry to make significant use of mathematics is not being fully met. The amount of time the majority of students spend on the study of mathematics in school in no way matches the importance of mathematical understanding to their lives, now and in the future.

Polls show that the public recognizes that mathematics is an essential subject for all students. Yet it is typical to find that only one year of mathematics is required for students in grades 9–12. Surveys of instructional time indicate that only a small percentage of time in elementary school is devoted to mathematics. And only a very tiny portion of that time is spent on teaching children to apply mathematics and solve problems.

When a student discontinues the study of mathematics early in high school, he or she is foreclosing on many options. Many doors, both in college programs and in vocational training, are at once closed to that person. These facts should be communicated more effectively to both the students and their parents.

The recommendation for increasing mathematics study imposes a special responsibility on professional leadership in mathematics education and the schools. To say that most students should study more mathematics is not to say that it should be the same mathematics for all. It does not mean simply keeping all students longer in the same traditional track. In fact, such a recommendation poses a tremendous challenge to curriculum developers and school districts to devise a more flexible range of options, a diversified program to meet a variety of interests, abilities, and goals.

Except for a distinction among courses typically taken in the ninth grade, differentiation in students' mathematics programs has been based mainly on whether or not a student takes elective years of mathematics. If everyone now needs more mathematics study, then differentiation must occur within the program.

The growing diversification of the applications of mathematics in an ever-increasing variety of college programs of study demands more than a single college-preparatory program. At the college level, the trend is to broaden the conception of mathematical study to include what is called "the mathematical sciences." Roughly, this means not only mathematics but the delivery of

mathematical ideas and tools to the solution of real-life problems. Present high school programs do not fully anticipate the many options provided by mathematics, the mathematical sciences, and computer science.

Technical and vocational training at different levels also assumes more and diverse mathematical backgrounds. For those whose formal education will end with high school, the needs of citizen and consumer for increasing mathematical sophistication dictate a collection of courses based on consumer and career needs, computer literacy, and quantitative literacy.

It is important that recommended programs permit lateral movement and not strictly "track" students, trapping them in a linear pattern that does not permit change to another path. Flexibility is vital, and the key is to keep options open as long as possible.

Since a higher level of mathematical skill and understanding will increasingly become a significant advantage in nearly all lives, then justice demands that all groups have equal access to these advantages. At present, females and some minority groups are underrepresented in mathematics courses and courses for which mathematics is prerequisite.

It is naive to suppose that just providing mathematics courses as electives will serve for equality of opportunity. All reasonable means should be employed to assure that everyone will have the foundation of mathematical learning essential to fulfilling his or her potential as a productive citizen. The currently underrepresented groups should be especially encouraged and helped.

Recognizing diversified individual interests and needs entails devising programs that are tailored for particular categories of students. Differentiated curricula must incorporate the special needs in mathematics of students with handicaps, including physical or learning difficulties. These programs will need to move away from the idea that everyone must learn the same mathematics and develop the same skills. Mathematics and mathematical ability cover a much broader range than most people realize.

In many current programs, the student who does poorly in the algorithmic skills finds progress in all aspects of mathematical development halted, since remediation is designed to concentrate solely on this deficiency. Remedial programs should identify other areas of mathematical ability—for example, spatially related skills—and concentrate attention also on the students' strengths, not solely on their deficits.

The student most neglected, in terms of realizing full potential, is the gifted student of mathematics. Outstanding mathematical ability is a precious societal resource, sorely needed to maintain leadership in a technological world.

Mathematics educators and curriculum developers should redesign the sequence of the curriculum to realize the critical process goals of problem solving, as well as content and skill goals. A clear and logical developmental sequence for process objectives from kindergarten through twelfth grade should be described and serve as an organizing framework.

In very general terms, such a sequence should proceed through stages of development, though in practice, progression should be smooth and unbroken.

In the elementary school, children, in their experiences in solving particular problems, gradually develop their higher-order mental abilities and learn particular skills and particular strategies. At this stage, strategies are primarily specific to individual problems. A foundation of skills must be laid, but skills should not be learned entirely in isolation from application even in the primary grades. There should be an interplay of skill building and application throughout.

Moving through the upper grades and into the junior high school level, the progress should be toward more generalization, more abstraction of techniques, more emphasis on similarities and patterns found in differing contexts. Strategies become not just ways to solve one type of problem but generalized and synthesized. Techniques learned in one context may be recognized as applicable to other problems. Specific attention is needed to help students make the important transition to the abstract reasoning processes.

During the seventh and eighth grades, intensive focus on problem solving should become a vehicle to exercise, confirm, and develop further all basic skills. At the same time, familiarity, competence, and confidence should be built in applying these mathematical skills to solving problems of varying difficulty and from diverse settings. At this stage, a significant skill is the ability to select strategies from a growing repertoire.

A broad range of problem-solving approaches should be explored so that the teacher can identify students' special strengths and advise them on high school options.

The ability to create strategies to attack a new problem is, at the junior high school level, simple and embryonic. It should increase in sophistication throughout high school. More flexibility and power should be achieved by more generalization and abstraction. The essential nature of techniques and strategies and their range of applicability should be emphasized as students begin to see more applications to broad categories of problem areas, as in, for example, application to other disciplines. There should be a greater effort made to coordinate the mathematics learned in mathematics classes with other subjects that use mathematics. Students who perform well in mathematics classes often fail to see any relationship in what they are doing there to the mathematical techniques employed in classes in science and other subjects.

This does *not* suggest that the content of high school mathematics is either dictated by or limited to the range of mathematical techniques used in other subjects. The goal is to develop a more flexible, deeper, and broader problem-solving power, and this goes significantly beyond the formulas and recipes that have been traditionally applied to familiar problems.

At the same time, the important interplay and integration of mathematics and its applications in learning should not cease because isolated course structure separates mathematics from disciplines that apply it. The principles of learning must take priority over administrative convenience. It is likely that this coordination will have to be accomplished by voluntary cooperative efforts

on the part of mathematics teachers and the teachers of other subjects that apply mathematics, with a healthy mutual respect for the legitimate goals of both groups.

At the high school level many students can apply their problem-solving abilities not only to problems of daily life, to problems from other disciplines, but to serious mathematical problems themselves.

Recommended Actions

6.1 *School districts should increase the amount of time students spend in the study of mathematics.*

- At least three years of mathematics should be required in grades 9 through 12.
- The amount of time allocated to learning mathematics in elementary school should be increased. It should range from a minimum of about five hours a week in the primary grades to a minimum of about seven hours a week in the upper grades. Part of this time can be gained by a program that stresses the application of mathematical skills in other subjects, especially the sciences and the social studies.
- A clear delineation of what constitutes college-level and precollege mathematics should be made.
- Colleges should not award college credit for courses in which the level of content is that of high school mathematics. This practice encourages students not to elect mathematics in high school beyond the minimum required.

6.2 *In secondary school, the curriculum should become more flexible, permitting a greater number of options for a diversified student population.*

- Increasing high school requirements in mathematics should not result in keeping all students longer in the same traditional tracks. These recommendations cannot be met with just the two-track alternatives of either general mathematics courses or precalculus courses typical of many existing programs.
- The high school curriculum should provide differing student populations with those appropriately organized areas of mathematical competence required by their needs, talents, and future objectives, but all presented with continual attention to functional problem-solving ability.
- Algebra should be included in the program of all capable students to keep their options open.
- For many students, algebra should be delayed until a level of maturity and basic mathematical understanding permit their taking full advantage of a significant algebra course. For many, this may not be ninth grade but perhaps eleventh or even twelfth grade. Significant

mathematics courses should be available to these students in ninth and tenth grades, not just the traditional general mathematics review or prealgebra courses.

- If such recommendations are followed, a course providing progress beyond junior high school but paving the way for a useful experience in consumer mathematics and later algebra needs to be developed.
- Consumer mathematics should develop a broader quantitative literacy and should consist primarily of work in informal statistics, such as organizing and interpreting quantitative information.
- All high school students should have work in computer literacy, and the hands-on use of computers, and the applications of computers where possible and appropriate throughout their mathematics programs.
- All students who plan to continue their study of mathematics beyond high school or to use it extensively in technical work or training should be enrolled in mathematics courses throughout their last high school year.

6.3 *Mathematics educators and college mathematicians should reevaluate the role of calculus in the differentiated mathematics programs.*

- Emerging programs that prepare users of mathematics in nontraditional areas of application may no longer demand the centrality of calculus that has traditionally been demanded for all students. (The Mathematical Association of America's PRIME 80 conference raised questions about the role of calculus as the eventual touchstone that dictated all college preparatory mathematics.)
- In light of this reevaluation, colleges and high schools should reexamine the concept of advanced placement in mathematics. For some students, though fully capable, an advanced placement program restricted to calculus placement may not be the optimal alternative. If advanced placement in mathematics is encouraged, it should be a broader concept that includes options in other branches of the mathematical sciences.

6.4 *The curriculum that stresses problem solving must pay special heed to the developmental sequence best suited to achieving* process *goals, not just content goals.*

- From the earliest years, the basic mathematical tools should be acquired within the framework of usage and application, however simple the examples may be at the beginning levels.
- Since there are usually multiple approaches to all but the most trivial problems, not all students should be expected to proceed in the same way. Value should be placed on a thoughtful and productive approach, not solely on a single correct answer.

- Since elementary school children differ widely in maturation and intellectual development, the teacher should be prepared to value and reward different contributions made by different students to the solution of a common problem.
- Team efforts in problem solving should be commonplace in the elementary school classroom.
- At middle school and junior high school levels, the focus of the curriculum should be on more formal and more general problem-solving approaches and strategies themselves.
- At middle school and junior high school, instruction should stress the ability to apply techniques used in one situation to new and unfamiliar situations.
- At middle school and junior high school, instruction should stress the ability to select from a range of strategies and to create new strategies by combining known techniques.
- At middle school and junior high school, instruction should aid in the student's transition to more abstract reasoning.
- Difficulty with paper-and-pencil computational algorithms should not interfere with the learning of problem-solving strategies.
- At the junior high school level, calculators should be available so that no student will be excluded from the opportunity to develop these strategies.
- All courses in high school mathematics should include some activities in applications.
- Teachers of mathematics and teachers of other disciplines should cooperate in assuring that students perceive the relationship of the mathematics they learn to the mathematics applied in problems in those other disciplines.
- Qualified mathematics teachers should be used as resource specialists for instructional programs in which mathematics methods are applied in other subjects.
- Teachers of mathematics should be prepared in the application of known problem-solving techniques to a variety of problems.

6.5 *Teachers, school officials, counselors, and parents should encourage a positive attitude toward mathematics and its value to the individual learner.*

- A curriculum that focuses on problem solving should challenge all students. It should present the opportunity for students of all ability levels to make a contribution, and it should promote the attitude that they are capable of solving problems.
- Programs that will encourage a larger percentage of females and

minority students to study more mathematics should be designed and supported.

- Parents and counselors should help students to recognize the importance of mathematics study to their futures and guide them to make appropriate school decisions.

6.6 *Special programs stressing problem-solving skills should be devised for special categories of students.*

- The professionals in mathematics and in special education should work together to identify the process abilities, possible and optimal, for students with handicaps and learning difficulties.
- Perception of what mathematical ability encompasses should be broadened beyond the linear, algorithmic stereotype. The significance of spatial perception and spatial relationships in problem solving needs to be stressed, for many students who do not fare well in algorithmic thinking may have special abilities that are spatial and geometric.
- Increased attention should be paid to developing the potential of the gifted student of mathematics.
- Colleges and schools should cooperate in devising imaginative programs for the mathematically gifted.
- In general, programs for the gifted should be based on a sequential program of enrichment through more ingenious problem-solving opportunities rather than through acceleration alone.
- Materials and resources of a sophistication and depth suitable to the unusual potential of the gifted student in problem solving should be developed.

Recommendation 7

MATHEMATICS TEACHERS MUST DEMAND OF THEMSELVES AND THEIR COLLEAGUES A HIGH LEVEL OF PROFESSIONALISM

Educators can gain the support of society and the rewards of a truly effective performance by developing, defining, and enforcing professional standards in terms of highly competent professional performance rather than by any other norm. This must be done to provide the nation, its young people, and its future with the mathematics programs worthy of them and of that future.

Within mathematics teaching—

—there are already many well-prepared and effective teachers who provide outstanding professional leadership;

—there are also many teachers who are motivated and desire to improve but who lack adequate preparation and must be given the necessary support to become fully qualified and to improve;

—there are, however, some teachers whose attitudes and functioning are at less than a professional level. In the best interests of students and society, the number of such teachers must be reduced immediately.

Even the best prepared, competent, and dedicated teachers must continue their development to keep abreast of changing needs, tools, and conditions. School administrations have the responsibility to make this possible by providing continuing in-service education and by encouraging teachers to take full advantage of opportunities for maintaining their competence that are offered by professional organizations and universities.

At the beginning of this decade, the schools are faced with a widespread shortage of qualified mathematics teachers. The demand for mathematical competence in many sectors of society is great and growing, and schools find it impossible to compete for individuals who have this desired background. Thus, in many mathematics classrooms, the teacher does not have the subject-matter qualifications for teaching mathematics. School administrations have two obligations: (1) to be forthright and open with the parents about the situation, and (2) to provide special aid and support needed by these teachers until they can make up deficiencies. The professional organizations also have a special responsibility to cooperate in assisting such teachers who are dedicated to improvement. However, the public and its representatives must give high priority to finding ways to solve the worsening problem.

Regardless of preparation, the standard of professionalism should be consistently high, and it is the obligation of any group that wishes to be called a profession to insist that all members maintain this standard.

Teachers must be sensitive to the needs of their students and dedicate themselves to the improvement of student learning as their primary professional objective. The right of students and parents to expect this dedication has correlative responsibilities: a mutual respect and support by the parent of the educational program and a teacher's professional competence, and the acceptance by the student of ultimate personal and active commitment to his or her own learning.

Any teacher who lacks dedication to these professional ideals and to continued self-improvement should not be retained in teaching. Teachers must accept performance and not protectionism as a synonym for professionalism.

During the decade of the 1980s, the continuing appearance of new concepts and theories in mathematics, in the applications of mathematics, and in the teaching-learning process will affect both curriculum and instruction in school mathematics. In order to remain professional, teachers must continue to study in all three areas. This will require of teachers a new level of motivation and dedication.

Recommended Actions

7.1 *Every mathematics teacher should accept responsibility for maintaining teaching competence.*

- Full advantage should be taken of all existing opportunities for continuing education.
- Teachers should insist that school districts and colleges make provision for in-service education and staff development opportunities.
- Teachers should belong to professional organizations that are dedicated to the improvement of teaching and learning.
- Teachers should participate actively in the efforts of professional organizations to improve teaching and learning.
- Teachers should share ideas and participate with their peers in cooperative efforts at self-improvement, including observation and constructive criticism of one another.
- Teachers as a profession should insist that all members maintain a consistently high standard of professional behavior. The profession is not obligated to protect those individuals who refuse to live up to reasonable professional standards.
- Bargaining units of organizations representing teachers should include in their requests release time for professional development and attendance at professional conferences that provide in-service education.

7.2 *School boards and school administrations should take all possible means to assure that mathematics programs are staffed by qualified, competent teachers who remain current in their field.*

- Necessary incentives must be found to attract competent and dedicated teachers to the profession.
- The status, compensation, and teaching conditions necessary for the retention of qualified teachers must be dramatically improved.
- School districts must budget adequately and provide incentives for teachers to participate in in-service education pertinent to their immediate needs as they prepare to meet the challenges of the future.
- School district staffs, including teachers, must plan in-service education that is articulated with local colleges and universities as well as with professional organizations such as the NCTM and its state and local affiliates.
- School administrators should encourage teachers to take an active professional role and should permit them to participate, without penalty, in conferences and vital professional work.
- School systems should maintain well-qualified mathematics specialists or supervisors at all levels to help teachers achieve the pro-

fessional level specified in these recommendations and to coordinate mathematics education efforts within the system.

- Well-qualified mathematics specialists should be on the staffs of all governmental agencies that deal with mathematics education—specifically, in all state and provincial departments of education, in the National Education Department, the National Institute of Education, and the National Science Foundation.

7.3 *Teacher education institutions and agencies should develop new programs of preparation to incorporate the problem-solving emphases recommended.*

- Programs in teacher education must be designed to prepare teachers for new levels of performance and professionalism.
- The effective teaching of problem solving requires thorough preparation both in mathematical content and in teaching methods that develop problem-solving ability.
- Colleges and universities must redesign courses and programs to incorporate revisions recommended for the preparation of mathematics teachers.

7.4 *Certification standards for mathematics teaching should be revised and upgraded to incorporate the needs reflected in these recommendations.*

- Professional organizations should participate in defining standards for certification of teachers.
- State and provincial departments of education should involve professional organizations in setting professional standards and defining qualifications.
- Professional organizations—such as the Mathematical Association of America, Conference Board of the Mathematical Sciences, National Council of Supervisors of Mathematics—in cooperation with the NCTM should continually review and update their guidelines for the preparation of teachers of mathematics.

Recommendation 8

PUBLIC SUPPORT FOR MATHEMATICS INSTRUCTION MUST BE RAISED TO A LEVEL COMMENSURATE WITH THE IMPORTANCE OF MATHEMATICAL UNDERSTANDING TO INDIVIDUALS AND SOCIETY

Solutions to the problems identified in previous sections cannot be achieved solely within the education community but require active participation and support by parental and societal groups.

The mathematics teaching profession recognizes and respects the right of parents and society to hold it accountable for the mathematical competence of children. However, in calling for particular programs of action, parents and society often mistakenly promote activities that are counterproductive to the realization of the goals they support. Communication and cooperation must bridge this gap.

The rededication of teachers called for in Recommendation 7 will be meaningless or impossible if society is not committed to supporting the professionalism it rightfully demands. Today, teachers' financial incentives are neither commensurate with the responsibilities they carry nor adequate to attract and retain them in a demanding, crucial, and sometimes burdensome profession. Few school systems have truly adequate supervisory and material support for the teacher in the maintenance and improvement of the instructional environment. More and more, teachers feel the lack of parental understanding of the complexity of their task and the lack of parental cooperation and support in their efforts to instruct children. Furthermore, governmental support for improving the quality of mathematics teaching has dwindled.

Essential to the success of the program outlined in these recommendations are (1) the willingness of government and private funding agencies to listen and respond more sensitively to the professionals in elementary and secondary mathematics education, and (2) the reestablishment by such agencies of institutes and other programs for the continuing education and upgrading of teachers.

At present, there are too many unnecessary obstacles to the effective functioning of teacher and student in a true teaching/learning interaction. These include more and more time required for unproductive record keeping; many unmotivated, undisciplined students; a lack of parental support; ambivalence and vacillation in government regulations; shifting societal priorities; the lack of home and school agreement on out-of-school study assignments.

There are also obstacles faced by school administrators and by the policy-making bodies, the school boards, whose ultimate responsibility is the effectiveness of the school district. It is recognized that in many instances confusions of legalisms and court decisions impose constraints on the ability of these people to accomplish what they agree should be done. Society today is troubled with many complexities. All institutions are threatened. These obstacles should not be an excuse for inaction; rather, in the light of pressing student needs it is imperative that they be surmounted.

All these things make the ideal professional role of the teacher almost impossible; they make it difficult to attract and retain teachers of a level of competence necessary to realize the outlined program. The immediate loser is the individual student, who is excluded from full participation in, and contribution to, society. The ultimate loser is society itself.

The public and its representatives need to confront a serious problem that is increasing in magnitude. There are not enough qualified mathematics teachers to fill mathematics classroom positions. Present enrollments in teacher

preparation programs guarantee that the shortage will increase. Mathematics is of critical importance and should be taught by those well qualified and knowledgeable. Such knowledge is not gained in a crash program of short duration. Parents concerned for their children's future and society concerned for its own future should immediately find extraordinary measures to solve this problem.

The professional community and society share a common goal: to bring all citizens to the full realization of their mathematical capacity. This is a complex and delicate task, and it requires the commitment and cooperation of all segments of society, not just the school, parents, and teachers.

Recommended Actions

8.1 *Society must provide the incentives that will attract and retain competent, fully prepared, qualified mathematics teachers.*

- School districts must provide compensation commensurate with the professionalism and qualifications necessary to achieve our educational goals in mathematics.
- School districts should investigate a variety of incentives and conditions that will stop the drain of qualified mathematics teachers to other, more highly compensated fields of work.
- School districts should assure mathematics teachers a classroom environment and conditions conducive to effective teaching, including a reasonable class size.
- School districts should provide teaching conditions and incentives that will attract dedicated and competent people into the mathematics teaching profession.

8.2 *Parents, teachers, and school administrations must establish new and higher standards of cooperation and teamwork toward the common goal of educating each student to his or her highest potential.*

- Professionals must respond to calls for maintaining educational standards and must work cooperatively with parents to specify these standards.
- Parents must support the maintenance of agreed on standards of achievement and discipline.
- Programs in mathematics that take full advantage of home and school cooperation should be systematically developed.
- Parents should enter the process of determining educational goals as partners, with shared participation and responsibility for the accomplishment of those goals.
- Parents should support the teacher's assignment of homework when it is reasonable and clearly related to the educational objectives.

- Parents should help guide students to an understanding of their critical need to learn mathematics.
- Parents should help students understand that they have the ultimate responsibility for their own learning and must be active and cooperative participants in the process.
- Parents and teachers should cooperate in a mutually supportive attack on the erosion of respect for authority in the classroom.
- Parents and teachers should cooperate in a mutually supportive attack on the erosion of motivation toward academic achievement.

8.3 *Government at all levels should operate to facilitate, not dictate, the attainment of goals agreed on cooperatively by the public's representatives and the professionals.*

- Government funding agencies should support an emphasis on research and development in applying mathematics to problem solving.
- Teachers should have advisory roles in all decisions of policy and support.
- When legislation affecting education is required, it should be entered into with caution and only after the involvement of educational professionals in formulating and reviewing the mathematical and pedagogical aspects of such legislation.
- Legislation concerning accountability should take into account the multiple factors that determine school achievement.
- Legislators should avoid tendencies to mandate testing as the sole criterion for the evaluation of educational success.
- Mandates for the achievement of minimal competencies should not limit the school mathematics program in its broader range of essential goals.
- Legislation should not determine educational goals but when necessary should facilitate the achievement of cooperatively agreed on objectives.
- Dual respect and effective articulation must become commonplace between the civic leaders who appropriately call for educational accountability and the educational professionals who must formulate responses.

National Council of Supervisors of Mathematics Position Paper on BASIC SKILLS

Introduction

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;
2. Reactions to the results of the National Assessment of Educational Progress;
3. Rising costs of education and increasing demands for accountability;
4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives;
5. Increased awareness of the need to provide remedial and compensatory programs;
6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations, and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a

TEN BASIC SKILL AREAS

Problem solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

Applying mathematics to everyday situations

The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

Alertness to the reasonableness of results

Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

Estimation and approximation

Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

Appropriate computational skills

Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

Geometry

Students should learn the geometric concepts will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

Measurement

As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

Reading, interpreting, and constructing tables, charts, and graphs

Students should know how to read and draw conclusions from simple tables, maps, charts, and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

Using mathematics to predict

Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

Computer literacy

It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

position on basic mathematical skills. This statement is the result of that effort.

Rationale for the Expanded Definition

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting, and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

Defining Basic Skills

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use handheld calculators and other electronic devices in mathematics.

Minimum Essentials for High-School Graduation

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high-school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills." If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematical skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high-school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

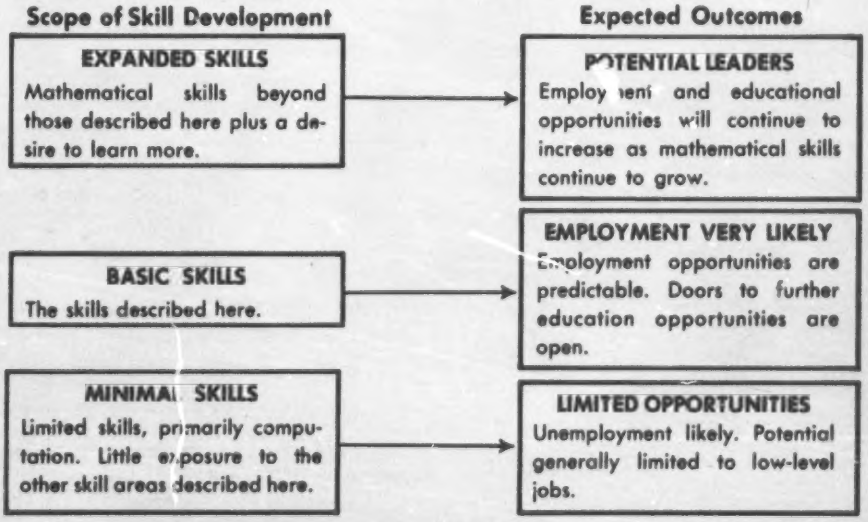
Developing the Basic Skills

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully under-

Basic Skills and the Student's Future

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.


Scope of Skill Development
Expected Outcomes
EXPANDED SKILLS
Mathematical skills beyond those described here plus a desire to learn more.
POTENTIAL LEADERS
Employment and educational opportunities will continue to increase as mathematical skills continue to grow.
BASIC SKILLS
The skills described here.
EMPLOYMENT VERY LIKELY
Employment opportunities are predictable. Doors to further education opportunities are open.
MINIMAL SKILLS
Limited skills, primarily computation. Little exposure to the other skill areas described here.
LIMITED OPPORTUNITIES
Unemployment likely. Potential generally limited to low-level jobs.

stand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle.

The learning of basic mathematical skills is a continuing process which extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

Evaluating and Reporting Student Progress

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

a. Items are not necessarily generated to measure a specific objective or instructional aim.

b. The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But, while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

Conclusion

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not "back" but *forward* to the essential needs of adults in the present and future.

The National Council of Supervisors of Mathematics (NCSM) is an organization of persons whose principal responsibility is the direction and supervision of the mathematics program in a public or nonpublic school district or school system. It is one of the affiliated groups of the NCTM.

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